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MATHEMATICS
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SEQUENCE & SERIES

THEORY AND EXERCISE BOOKLET

CONTENTS

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JEE Syllabus :

Arithmetic, geometric and harmonic progressions, arithmetic, geometric and harmonic means, sums of finite arithmetic and geometric progressions, infinite geometric series, sums of squares and cubes of the first n natural numbers.

A. SEQUENCE

A sequence is a set of terms in a definite order with a rule for obtaining the terms.
e.g. 1 , 1/2 , 1/3 , ……. , 1/n , …….. is a sequence .
A sequence is a function whose domain is the set N of natural numbers. Since the domain for every sequence is the set of N natural numbers, therefore a sequence is represented by its range. $f : N \to R$, then $f(n) = t_n$ $n \in N$ is called a sequence and is denoted by
$\{f(1), f(2), f(3), \ldots\ldots\ldots\ldots\} =$
$\{t_1, t_2, t_3, \ldots\ldots\ldots\ldots\} = \{t_n\}$

Real Sequence : A sequence whose range is a subset of R is called a real sequence.
Examples :
(i) 2, 5, 8, 11, ………. (ii) 4, 1, –2, –5, ……….. (iii) 3, –9, 27, –81, …………..

Types of Sequence : On the basis of the number of terms there are two types of sequence.
(i) Finite sequence : A sequence is said to be finite if it has finite number of terms.
(ii) Infinite sequence : A sequence is said to be infinite if it has infinite number of terms.

Series : By adding or substracting the terms of a sequence, we get an expression which is called a series. If $a_1, a_2, a_3, \ldots\ldots, a_n$ is a sequence, then the expression $a_1 + a_2 + a_3 + \ldots\ldots + a_n$ is a series.
Example. (i) $1 + 2 + 3 + 4 + \ldots\ldots\ldots + n$ (ii) $2 + 4 + 8 + 16 + \ldots\ldots\ldots$

Progression : It is not necessary that the terms of a sequence always follow a certain pattern or they are described by some explicit formula of the n^{th} term. Those sequences whose terms follow certain patterns are called progressions.

Ex.1 Write down the sequence whose nth term is **(i)** $(-1)^n\left(\dfrac{3n+2}{5}\right)$ **(ii)** $\dfrac{1}{n^2}\sin\left(\dfrac{n\pi}{3}\right)$

Sol. **(i)** Let $a_n = (-1)^n\left(\dfrac{3n+2}{5}\right)$. Putting = 1, 2, 3, 4,…… successively, we get

$a_1 = (-1)^1\left(\dfrac{3.1+2}{5}\right) = -1$ $a_2 = (-1)^2\left(\dfrac{3.2+2}{5}\right) = 8/5$

$a_3 = (-1)^3\left(\dfrac{3.3+2}{5}\right) = -11/5$ $a_4 = (-1)^4\left(\dfrac{3.4+2}{5}\right) = 14/5$

……………………………………
Hence we obtain the sequence –1, 8/5, –11/5, 14/5,……

(ii) Let $a_n = \dfrac{1}{n^2}\sin\left(\dfrac{n\pi}{3}\right)$. Putting n = 1, 2, 3, 4,... successively, we get

$a_1 = \dfrac{1}{1^2}\sin\left(\dfrac{\pi}{3}\right) = \sqrt{3}/2$ $a_2 = \dfrac{1}{2^2}\sin\left(\dfrac{2\pi}{3}\right) = \sqrt{3}/8$

$a_3 = \dfrac{1}{3^2}\sin\left(\dfrac{3\pi}{3}\right) = 0$ $a_4 = \dfrac{1}{4^2}\sin\left(\dfrac{4\pi}{3}\right) = -\sqrt{3}/32$

……………………………………
Hence we obtain the sequence $\sqrt{3}/2, \sqrt{3}/8, 0, -\sqrt{3}/32, \ldots$

Ex.2 If sum of n terms of a sequence is given by $S_n = 2n^2 + 3n$, find its 50^{th} term.

Sol. Let t_n is n^{th} term of the sequence so $t_n = s_n - s_{n-1} = 2n^2 + 3n - 2(n-1)^2 - 3(n-1) = 4n + 1$
so $t_{50} = 201$.

B. ARITHMETIC PROGRESSION (AP)

AP is a sequence whose terms increase or decrease by a fixed number . This fixed number is called the common difference . If a is the first term & d the common difference, then AP can be written as a, $a + d$, $a + 2d$, ... $a + (n-1)d$, ...

n^{th} term of this AP $t_n = a + (n-1)d$, where $d = a_n - a_{n-1}$.

The sum of the first n terms of the AP is given by ; $S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}[a + l]$.

where l is the last term.

Remarks :

(i) If each term of an A.P. is increased, decreased, multiplied or divided by the same non zero number, then the resulting sequence is also an AP .

(ii) Three numbers in AP can be taken as $a - d$, a, $a + d$; four numbers in AP can be taken as $a - 3d$, $a - d$, $a + d$, $a + 3d$; five numbers in AP are $a - 2d$, $a - d$, a, $a + d$, $a + 2d$ & six terms in AP are $a - 5d$, $a - 3d$, $a - d$, $a + d$, $a + 3d$, $a + 5d$ etc.

(iii) The common difference can be zero, positive or negative .

(iv) The sum of the terms of an AP equidistant from the beginning & end is constant and equal to the sum of first & last terms .

(v) Any term of an AP (except the first) is equal to half the sum of terms which are equidistant from it . $a_n = 1/2\,(a_{n-k} + a_{n+k})$, $k < n$.
For $k = 1$, $a_n = (1/2)(a_{n-1} + a_{n+1})$; For $k = 2$, $a_n = (1/2)(a_{n-2} + a_{n+2})$ and so on .

(vi) $t_r = S_r - S_{r-1}$ **(vii)** If a, b, c are in AP $\Rightarrow 2b = a + c$.

Ex. 3 Find the sum of all the three digit natural numbers which on division by 7 leaves remainder 3.

Sol. All these numbers are 101, 108, 115,....... 997, to find n.

$997 = 101 + (n-1)\,7 \Rightarrow n = 129$ so $S = \frac{129}{2}[101 + 997] = 70821$.

Ex. 4 The sum of first three terms of an A.P. is 27 and the sum of their squares is 293 . Find the sum to 'n' terms of the A.P.

Sol. Let $a - d$, a, $a + d$ be the numbers $\Rightarrow a = 9$
Also $(a - d)^2 + a^2 + (a + d)^2 = 293$. $\Rightarrow 3a^2 + 2d^2 = 293 \Rightarrow d^2 = 25 \Rightarrow d = \pm 5$
therefore numbers are 4, 9, 14.

Hence the A.P. is 4, 9, 14, ... $\Rightarrow s_n = \frac{n}{2}[5n + 3]$ or 14, 9, 4, ... $\Rightarrow s_n = \frac{n}{2}[33 - 5n]$

Ex.5 Let a_n be the n^{th} term of an arithmetic progression. Let S_n be the sum of the first n terms of the arithmetic progression with $a_1 = 1$ and $a_3 = 3a_8$. Find the largest possible value of S_n.

Sol. From $a_3 = 3a_8$ we obtain $1 + 2d = 3(1 + 7d) \Rightarrow d = -\frac{2}{19}$.

Then $S_n = \frac{n}{2}\left(2-(n-1)\frac{2}{19}\right) = \frac{n}{19}[19 - (n - 1)] = \frac{(20-n)n}{19}$.

now consider $20n - n^2 = -[n^2 - 20n] = -[(n - 10)^2 - 100]$

$\therefore$ $S_n = \frac{100-(n-10)^2}{19}$ now, S_n will be maximum if n = 10 and $(S_n)_{max} = \frac{100}{19}$

Ex.6 Suppose $a_1, a_2, \ldots$ are in A.P. and S_k denotes the sum of the first k terms of this A.P. If $S_n/S_m = n^4/m^4$ for all m, n, $\in$ **N**, then prove that $\frac{a_{m+1}}{a_{n+1}} = \frac{(2m+1)^3}{(2n+1)^3}$

Sol. Putting $a_1 = a$, we have $\frac{S_n}{S_m} = \frac{n[2a+(n-1)d]/2}{m[2a+(n-1)d]/2} = \frac{n^4}{m^4} \Rightarrow \frac{2a+(n-1)d}{2a+(m-1)d} = \frac{n^3}{m^3}$

Replacing n by 2n + 1 and m by 2m + 1, we get

$$\frac{2a+(2n+1-1)d}{2a+(m+1-1)d} = \frac{(2n+1)^3}{(2m+1)^3} \Rightarrow \frac{a_{n+1}}{a_{m+1}} = \frac{(2n+1)^3}{(2m+1)^3} \Rightarrow \frac{a_{m+1}}{a_{n+1}} = \frac{(2m+1)^3}{(2n+1)^3}$$

Ex.7 If the sum of m terms of an A.P. is equal to the sum of either the next n terms or the next p terms prove that $(m + n)\left(\frac{1}{m}-\frac{1}{p}\right) = (m + p)\left(\frac{1}{m}-\frac{1}{n}\right)$

Sol. Let the A.P. be a, a + d, a + 2d,....

Given $T_1 + T_2 + \ldots + T_m = T_{m+1} + T_{m+2} + \ldots + T_{m+n}$ ...(1)

Adding $T_1 + T_2 + \ldots + T_m$ on both sides in (i), Then

$2(T_1 + T_2 + \ldots + T_m) = T_1 + T_2 + \ldots + T_m + T_{m+1} + \ldots + \ldots + T_{m+n}$ $\Rightarrow$ $2S_m = S_{m+n}$

$\therefore$ $2.\frac{m}{2}\{2a + (m - 1)d\} = \frac{m+n}{2}\{2a + (m + n - 1)d\}$

Let $2a + (m - 1)d = x$ $\Rightarrow$ $mx = \frac{m+n}{2}\{x + nd\}$ ...(2)

$\Rightarrow$ $(m - n)x = (m + n)nd$ again $T_1 + T_2 + \ldots + T_m = T_{m+1} + T_{m+2} + \ldots + T_{m+p}$ ...(3)

Similarly $(m - p)x = (m + p)pd$

dividing (2) by (3), we get $\frac{m-n}{m-p} = \frac{(m+n)n}{(m+p)p}$ $\Rightarrow$ $(m - n)(m + p)p = (m - p)(m + n)n$

dividing both sides by mnp, we have $(m + p)\left(\frac{1}{n}-\frac{1}{m}\right) = (m + n)\left(\frac{1}{p}-\frac{1}{m}\right)$

Hence $(m + n)\left(\frac{1}{p}-\frac{1}{m}\right) = (m + p)\left(\frac{1}{m}-\frac{1}{n}\right)$

Ex.8 Show that any positive integral power (except the first) of a positive integer m, is the sum of m consecutive odd positive integers. Find the first odd integer for m^r ($r > 1$).

Sol. Let us find k such that $m^r = (2k + 1) + (2k + 3) + ... + (2k + 2m - 1)$

$$\therefore \quad m^r = \frac{m}{2}[2k + 1 + 2k + 2m - 1] \quad \Rightarrow \quad m^{r-1} = 2k + m$$

Note that $m^{r-1} - m$ is an even integer for all r, m $\in$ **N** and $r > 1$. Therefore, $k = (m^{r-1} - m)/2$ is an integer. Thus, the first term is given by $m^{r-1} - m + 1$.

C. GEOMETRIC PROGRESSION (GP)

GP is a sequence of numbers whose first term is non zero & each of the succeeding terms is equal to the proceeding terms multiplied by a constant . Thus in a GP the ratio of successive terms is constant. This constant factor is called the **COMMON RATIO** of the series & is obtained by dividing any term by that which immediately proceeds it . Therefore a, ar, ar^2, ar^3, ar^4, ... is a GP with a as the first term & r as common ratio .

(i) n^{th} term = $a\,r^{n-1}$ **(ii)** Sum of the I^{st} n terms i.e. $S_n = \frac{a(r^n - 1)}{r-1}$, if $r \neq 1$.

(iii) Sum of an infinite GP when $|r| < 1$ when $n \to \infty$ $r^n \to 0$ if $|r| < 1$ therefore, $S_\infty = \frac{a}{1-r}\ (|r| < 1)$.

(iv) If each term of a GP be multiplied or divided by the same non-zero quantity, the resulting sequence is also a GP .

(v) Any 3 consecutive terms of a GP can be taken as a/r, a, ar ; any 4 consecutive terms of a GP can be taken as a/r^3, a/r, ar, ar^3 & so on.

(vi) If a, b, c are in GP $\Rightarrow b^2 = ac$.

Ex. 9 If the first term of G.P. is 7, its n^{th} term is 448 and sum of first n terms is 889, then find the fifth term of G.P.

Sol. Given a = 7 the first term $t_n = ar^{n-1} = 7(r)^{n-1} = 448 \quad \Rightarrow \quad 7r^n = 448\,r$

Also $S_n = \frac{a(r^n - 1)}{r-1} = \frac{7(r^n - 1)}{r-1} \Rightarrow 889 = \frac{448r - 7}{r-1} \quad \Rightarrow \quad r = 2$. Hence $T_5 = ar^4 = 7(2)^4 = 112$

Ex.10 If the third and fourth terms of an arithmetic sequence are increased by 3 and 8 respectively, then the first four terms form a geometric sequence. Find

(i) the sum of the first four terms of A.P. **(ii)** second term of the G.P.

Sol. a, (a + d), (a + 2d), (a + 3d) in A.P.

a, a + d, (a + 2d + 3), (a + 3d + 8) are in G.P. hence a + d = ar

also $r = \frac{a+d}{a} = \frac{a+2d+3}{a+d} = \frac{a+3d+8}{a+2d+3}$

$$\therefore \quad \frac{d+3}{d} = \frac{d+5}{d+3} \quad \Rightarrow \quad d^2 + 6d + 9 = d^2 + 5d \quad \Rightarrow \quad d = -9$$

$$\therefore \quad \frac{a-9}{a} = \frac{a-15}{a-9} \quad \Rightarrow \quad a^2 - 18a + 81 = a^2 - 15a \quad \Rightarrow \quad 3a = 81 \quad \Rightarrow \quad a = 27$$

hence A.P. is 27, 18, 9, 0, G.P. is 27, 18, 12, 8

(i) sum of the first four terms of A.P. = 54 (ii) 2^{nd} term of G.P. = 18

Ex.11 Three positive numbers form a G.P. If the second term is increased by 8, the resulting sequence is an A.P. In turn, if we increase the last term of this A.P. by 64, we get a G.P. Find the three numbers .

Sol. Let the numbers be a, ar, ar^2 where $r > 0$

Hence a, (ar + 8), ar^2 in A.P. – (1) Also a, (ar + 8), $ar^2 + 64$ in G.P. — (2)

(2) $\Rightarrow$ $(ar + 8)^2 = a(ar^2 + 64)$ $\Rightarrow$ $a = \dfrac{4}{4-r}$ — (3)

Also (1) $\Rightarrow$ $2(ar + 8) = (a + ar^2)$ $\Rightarrow$ $(1 - r)^2 = \dfrac{16}{a}$ — (4)

From (3) and (4) r = 3 or – 5 (rejected) . Hence a = 4 numbers are : 4, 12, 36

Ex.12 The sum of the first five terms of a geometric series is 189, the sum of the first six terms is 381, and the sum of the first seven terms is 765. What is the common ratio in this series.

Sol. $S_5 = 189; S_6 = 381;$ $S_7 = 765$; $t_6 = S_6 - S_5 = 381 - 189 = 192$ $t_7 = S_7 - S_6 = 765 - 381 = 384$

now common ratio = $\dfrac{t_7}{t_6} = \dfrac{384}{192} = 2$

Ex.13 Three positive distinct numbers x, y, z are three terms of geometric progression in an order and the numbers x + y, y + z, z + x are three terms of arithmetic progression in that order. Prove that $x^x \cdot y^y = x^y \cdot y^z \cdot z^x$.

Sol. Let x be first term of G.P. and y and z be the m^{th} and n^{th} of same G.P. respectively

$\Rightarrow$ $t_m = y = xr^{m-1}$ and $t_n = z = x\,r^{n-1}$, where r is a common ratio of G.P. $\Rightarrow$ $\dfrac{m-1}{n-1} = \dfrac{\log(y/x)}{\mathrm{lo}(z/x)}$ ...(1)

Now, we have y + z = x + y + (m – 1) d and z + x = x + y + (n – 1) d, where id ia common difference of A.P. $\Rightarrow$ $\dfrac{m-1}{n-1} = \dfrac{z-x}{z-y}$ From (1) and (2), $\dfrac{\log(y/x)}{\log(z/x)} = \dfrac{z-x}{z-y}$

$\Rightarrow$ $\dfrac{y}{x} = \left(\dfrac{z}{x}\right)^{\frac{z-x}{z-y}}$ $\Rightarrow$ $y = z^{(z-x)/(z-y)} \cdot x^{(x-y)/(z-y)}$ $\Rightarrow$ $y^{(z-y)} = z^{(z-x)} \cdot x^{(x-y)}$ $\Rightarrow$ $x^x \cdot y^y \cdot z^z = x^y \cdot y^z \cdot z^x$

Ex.14 Using G.P. express $0.\overline{3}$ and $1.2\overline{3}$ as $\dfrac{p}{q}$ form.

Sol. Let x = $0.\overline{3}$ = 0.3333............ = 0.3 + 0.03 + 0.003 + 0.0003 + ...

$$= \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{100000} + \ldots\ldots\ldots\ldots\ldots = \frac{\frac{3}{10}}{1-\frac{1}{10}} + \frac{3}{9} = \frac{1}{3}.$$

Let y = $1.2\overline{3}$ = 1.233333 = 1.2 + 0.03 + 0.003 + 0.0003 +...............

$$= 1.2 + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \ldots\ldots\ldots\ldots\ldots = 1.2 + \frac{\frac{3}{10^2}}{1-\frac{1}{10}} = 1.2 + \frac{1}{30} = \frac{37}{30}.$$

Ex.15 The 1st, 2nd and 3rd terms of an arithmetic series are a, b and a^2 where 'a' is negative. The 1st, 2nd and 3rd terms of a geometric series are a, a^2 and b find the

(a) value of a and b

(b) sum of infinite geometric series if it exists. If no then find the sum to n terms of the G.P.

(c) sum of the 40 term of the arithmetic series.

Sol. a, b, a^2 are in A.P. $(a < 0)$ a, a^2, b are in G.P.

(a) $2b = a + a^2 \Rightarrow b = \frac{a}{2}(a + 1)$ and $a^4 = ab$, $a^3 = b$ since $a < 0$ $[\therefore a \neq 0]$

$a^3 = \frac{a}{2}(a + 1) \Rightarrow 2a^2 - a - 1 = 0 \Rightarrow (2a + 1)(a - 1) = 0$, a = 1 (rejected as $a < 0$) or $a = -\frac{1}{2}$

$\Rightarrow a^2 = \frac{1}{4}$; $\therefore$ $b = -\frac{1}{4}\left(-\frac{1}{2}+1\right) = -\frac{1}{8}$ $\therefore$ common difference $b - a = -\frac{1}{8} + \frac{1}{2} = \frac{3}{8}$

(b) common ratio $r = \frac{a^2}{a}$; $r = \frac{1\times 2}{4\times(-1)} = -\frac{1}{2}$; Since $|r| < 1$ $\therefore$ S_∞ exists

$$\therefore \quad S_\infty = -\frac{1}{2\left(1+\frac{1}{2}\right)} = -\frac{1}{3}$$

(c) $S_{40} = 20\left\{-1+39\times\frac{3}{8}\right\} = \frac{20}{8}\{-8+117\} = \frac{5}{2} \times 109 = \left\{-1+39\times\frac{3}{8}\right\} = \frac{545}{2}$

Ex.16 In an arithmetic progression, the third term is 15 and the eleventh term is 55. An infinite geometric progression can be formed beginning with the eighth term of this A.P. and followed by the fourth and second term. Find the sum of this geometric progression upto *n* terms. Also compute S_∞ if it exists.

Sol. Given $a + 2d = 15$ ....(1), $a + 10d = 55$ ....(2)

solving (1) and (2) $\therefore$ $8d = 40$ $d = 5$ & $a = 5$

$\therefore$ $t_8 = 40$ & $t_4 = 20$ $t_2 = 10$

hence the G.P. is 40, 20, 10, ... ∞ ($|r| < 1$) hence S_∞ exists.

$$\therefore \quad S_\infty = \frac{40}{1-(1/2)} = 80 \qquad S_n = \frac{40\left(1-\frac{1}{2^n}\right)}{1-\frac{1}{2}} = 80\left(1-\frac{1}{2^n}\right)$$

Ex.17 If the first 3 consecutive terms of a geometrical progression are the roots of the equation $2x^3 - 19x^2 + 57x - 54 = 0$ find the sum to infinite number of terms of G.P.

Sol. let the roots be $\frac{a}{r}$, a, ar

$\therefore$ $a\left(1+r+\frac{1}{r}\right)=\frac{19}{2}$ ....(1), $a^2\left(1+r+\frac{1}{r}\right)=\frac{57}{2}$ ....(2), $a^3\left(1+r+\frac{1}{r}\right)=\frac{54}{2}=27$ ....(3) $\therefore$ a = 3

from (1) $3(r^2 + r + 1) = \frac{19r}{2}$ $\Rightarrow$ $6r^2 + 6r + 6 = 19r$ $\Rightarrow$ $6r^2 - 13r + 6 = 0$

$\Rightarrow$ $(2r - 3)(3r - 2) = 0$ $\Rightarrow$ $r = \frac{2}{3}$ or $\frac{3}{2}$ (rejected)

$\therefore$ Numbers are $\frac{a}{r}$, a, ar $\Rightarrow$ $\frac{9}{2}$, 3, 2 $\qquad S_\infty = \frac{9/2}{1-(2/3)} = \frac{9\times 3}{2} = \frac{27}{2}$

Ex.18 Let $P = \prod_{n=1}^{\infty}\left(10^{\frac{1}{2^{n-1}}}\right)$ then find $\log_{0.01}(P)$.

Sol. $P = 10^1 \cdot 10^{\frac{1}{2}} \cdot 10^{\frac{1}{2^2}} \cdot 10^{\frac{1}{2^3}} \ldots\ldots\ldots\ldots = 10^{1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots\ldots+\infty} = 10^{\frac{1}{1-\frac{1}{2}}} = 10^2 = 100$ $\therefore$ $\log_{0.01}(P) = -1$

Ex.19 A positive number is such that its fractional part, its integral part and the number itself constitute the first three terms of a geometrical progression . Show that the n^{th} term and the sum of the first n terms of the G.P. are, $2^{n-2} \cdot \cos^{n-2}\frac{\pi}{5}$ and $2^n \cdot \cos^n\frac{\pi}{5} - 1$ respectively.

Sol. $x \begin{array}{l} m \text{ integral} \\ y \text{ fractional} \end{array}$ $\Rightarrow$ y, m, m + y in G.P. $m^2 = y(m + y)$

$\Rightarrow$ $y^2 + my - m^2 = 0$ If $y = \frac{\left(\sqrt{5}-1\right)m}{2}$ put m = 1 ; $y = \frac{\sqrt{5}-1}{2}$ or

$y = \frac{\sqrt{5}+1}{2}$ (not possible) G.P. $\frac{\sqrt{5}-1}{2}$, 1 , $\frac{\sqrt{5}+1}{2}$ .... Now proceed

Ex.20 Given a three digit number whose digits are three successive terms of a G.P. If we subtract 792 from it, we get a number written by the same digits in the reverse order . Now if we subtract four from the hundred's digit of the initial number and leave the other digits unchanged, we get a number whose digits are successive terms of an A.P. Find the number.

Sol. Let the number be $100x + 10y + z$ $\qquad y^2 = xz$ — (1)

$100x + 10y + z - 792 = 100z + 10y + z$ or $100(x - z) + (z - x) = 792$ $\Rightarrow$ $x - z = \frac{792}{99} = 8$

Also $2y = x - 4 + z$ $\Rightarrow$ $y = \frac{x+z-4}{2}$ — (2)

$\Rightarrow (x + z - 4)^2 = 4xz \Rightarrow (x + z)^2 - 8(x + z) + 16 = 4xz \Rightarrow (x - z)^2 - 8(x + z) + 16 = 0 \Rightarrow x + z = 10$

Ex.21 One of the roots of the equation $2000x^6 + 100x^5 + 10x^3 + x - 2 = 0$ is of the form $\frac{m+\sqrt{n}}{r}$, where m is non zero integer and n and r are relatively prime natural numbers. Find the value of m + n + r.

Sol. $2000x^6 + \underbrace{100x^5 + 10x^3 + x}_{\text{a G.P.}} - 2 = 0 \quad \Rightarrow 2000x^6 + \frac{x((10x^2)^3 - 1)}{10x^2 - 1} - 2 = 0$

$\Rightarrow \frac{x(1000x^6 - 1)}{10x^2 - 1} = -2(1000x^6 - 1) \quad \therefore 1000x^6 - 1 = 0$ or $\frac{x}{10x^2 - 1} = -2 \quad \Rightarrow \quad x = -(10x^2 - 1)$

$\Rightarrow x^2 = \frac{1}{10}$ which is not possible $20x^2 + x - 2 = 0 \quad \Rightarrow x = \frac{-1 \pm \sqrt{1+160}}{40} = \frac{-1+\sqrt{161}}{40}$ or $\frac{-1-\sqrt{161}}{40}$

$\therefore$ m = – 1; n = 161; r = 40 $\qquad$ m + n + r = 200

Ex.22 Suppose that $a_1, a_2, \ldots\ldots, a_n, \ldots\ldots$ is an A.P. Let $S_k = a_{(k-1)n+1} + a_{(k-1)n+2} + \ldots\ldots + a_{kn}$. Prove that $S_1, S_2, \ldots$ are in A.P. having common difference equal to n^2 times the common difference of the A.P. $a_1, a_2, \ldots$

Sol. We have $S_k = a_{(k-1)n+1} + a_{(k-1)n+2} + \ldots + a_{(k-1)n+n}$

Now we have $a_{(k-1)n+1} = a_1 + (k-1)nd$; $\quad a_{(k-1)n+2} = a_1 + (k-1)nd + d$

$a_{(k-1)n+3} = a_1 + (k-1)nd + 2d \quad \ldots\ldots\ldots\ldots$

$a_{(k-1)n=n} = a_1 + (k-1)d + \frac{(n-1)nd}{2}$ and $S_{k+1} = na + n^2kd + \frac{(n-1)nd}{2}$

[putting k + 1 in place of k] $\qquad$ Hence, we have $S_{k+1} - S_k = n^2d$

which is a constant independent of k. This proves that the sequence (S_k) is an A.P. having common difference n^2d.

Ex.23 In a G.P. if S_1, S_2 & S_3 denote respectively the sums of the first n terms, first 2n terms and first 3n terms, then prove that, $S_1^2 + S_2^2 = S_1(S_2 + S_3)$.

Sol. $S_1 = \frac{a(1-r^n)}{1-r}$; $\quad S_2 = \frac{a(1-r^{2n})}{1-r}$; $\quad S_3 = \frac{a(1-r^{3n})}{1-r}$

T P T $\qquad S_1(S_1 - S_3) = S_2(S_1 - S_2)$

L H S $\quad \frac{a(1-r^n)}{1-r}\left[\frac{a(1-r^n) - a(1-r^{2n})}{1-r}\right] = \frac{a^2(1-r^n)}{(1-r)^2}(r^{3n} - r^n) = \frac{a^2 r^n (1-r^n)(r^{2n}-1)}{(1-r)^2}$

Ex. 24 How many increasing 3-term geometric progressions can be obtained from the sequence $1, 2, 2^2, 2^3, \ldots, 2^n$? (e.g., $\{2^2, 2^5, 2^8\}$ is a 3-term geometric progression for $n \geq 8$.)

Sol. Let us start counting 3–term G.P.'s with common ratios $2, 2^2, 2^3, \ldots$

The 3–term G.P.'s with common ratio 2 are $1, 2, 2^2$; $2, 2^2, 2^3$; ....; $2^{n-2}, 2^{n-1}, 2^n$.

Theyy are (n – 1) in number. The 3–term GP's with common ratio 2^2 are

$1, 2^2, 2^4$; $2, 2^3, 2^5$; .... ; $2^{n-4}, 2^{n-2}, 2^n$

They are (n – 3) in number. Similarly we see that the 3–term GP's with common ratio 2^3 are (n – 5) in number and so on. Thus the number of 3–term GP's which can be formed from the sequence 1, 2, 2^2, 2^3,.....,2^n is equal to S = (n – 1) + (n – 3) + (n – 5) +......... .

Here the last term is 2 or 1 according as n is odd or even. If n is odd, then

$$S = (n-1) + (n-3) + (n-5) + ... + 2 = 2(1 + 2 + 3 + ... + \frac{n-1}{2}) \qquad = \frac{n^2-1}{4}.$$

If n is even, then $\quad S = (n-1) + (n-3) + ... + 1 = \frac{n^2}{4}$.

Hence the required number is $(n^2 - 1)/4$ or $n^2/4$ according as n is odd or even.

Ex.25 $N = 2^{n-1}(2^n - 1)$ and $(2^n - 1)$ is a prime number.

$1 < d_1 < d_2 < ..... < d_k = N$ are the divisors of N. Show that $\frac{1}{1} + \frac{1}{d_1} + \frac{1}{d_2} + ... + \frac{1}{d_k} = 2$.

Sol. Let $2^n - 1 = q$

1, d_1 d_2 .... d_k are divisors $\qquad$ 1, 2, 2^2, ..., 2^{n-1}, 2q, ..., 2^{n-1} q respectively.

$$\text{So, } S = \frac{1}{1} + \frac{1}{d_1} + \frac{1}{d_2} + ... + \frac{1}{d_k} \qquad = \frac{1}{1} + \frac{1}{2} + \frac{1}{2^2} + ... + \frac{1}{2^{n-1}} + \frac{1}{q}\left[\frac{1}{1} + \frac{1}{2} + \frac{1}{2^2} + ... + \frac{1}{2^{n-1}}\right]$$

$$\therefore \quad S = \frac{2^n - 1}{2^{n-1}} + \frac{1}{q}\frac{(2^n - 1)}{2^{n-1}} \qquad = \frac{(2^n-1)(q+1)}{q2^{n-1}} = \frac{(2^n-1)(2^n)}{(2^n-1)(2^{n-1})} = \frac{2^n}{2^{n-1}} = 2.$$

Ex.26 The first two terms of an arithmetic and a geometric progression with positive terms are equal. Prove that all other terms of the arithmetic progression are not greater other terms of the arithmetic progression are not greater than the corresponding terms of the geometric progression.

Sol. Let the first common term of the progression be a, and the second b. Then the nth term of the arithmetic progression will be equal to a + (b – a) (n – 1)

and the corresponding terms of the geometric progression has the form $a(b/a)^{n-1}$

And so, we have to prove that $a + (b - a)(n - 1) \le a(b/a)^{n-1}$.

in other words, that $a + (b - a)(n - 1) - a(b/a)^{n-1} \le 0$. or $\quad a\left\{\left(\frac{b}{a} - 1\right)(n-1) - \left[\left(\frac{b}{a}\right)^{n-1} - 1\right]\right\} \le 0.$

Let us rewrite the left member of this inequality as follows

$$a\left(\frac{b}{a} - 1\right)\left\{(n-1) - \left[\left(\frac{b}{a}\right)^{n-2} + \left(\frac{b}{a}\right)^{n-3} + ... + \left(\frac{b}{a}\right) + 1\right]\right\}.$$

Considering separately the three cases $\frac{b}{a} > 1, \frac{b}{a} < 1, \frac{b}{a} = 1$, we easily prove the validity of our inequality.

D. ARITHMETICO-GEOMETRIC SERIES

A series each term of which is formed by multiplying the corresponding term of an AP & GP is called the Arithmetico-Geometric Series . e.g. $1 + 3x + 5x^2 + 7x^3 + ...$
Here 1, 3, 5, ... are in AP & $1, x, x^2, x^3$ ..... are in GP .

Sum of n terms of an Arithmetico-Geometric Series

Let $S_n = a + (a + d)r + (a + 2d)r^2 + ... + [a + (n-1)d]\, r^{n-1}$

then $S_n = \dfrac{a}{1-r} + \dfrac{dr\left(1-r^{n-1}\right)}{(1-r)^2} - \dfrac{[a+(n-1)d]r^n}{1-r}$, $r \neq 1$.

Sum To Infinity : If $|r| < 1$ & $n \to \infty$ then $\underset{n\to\infty}{\text{Limit}}\, r^n = 0$. $S_\infty = \dfrac{a}{1-r} + \dfrac{dr}{(1-r)^2}$.

Ex.27 Find the sum to n terms of the series $\dfrac{1}{2}+\dfrac{2}{4}+\dfrac{3}{8}+\dfrac{4}{16}+\dfrac{5}{32}+........$ Also find the sum if it exist if $n \to \infty$.

Sol. $S = \dfrac{1}{2}+\dfrac{2}{4}+\dfrac{3}{8}+\dfrac{4}{16}+\dfrac{5}{32}+........+\dfrac{n}{2^n}$ ....(1) $\dfrac{S}{2} = \dfrac{1}{4}+\dfrac{2}{8}+\dfrac{3}{16}+.............+\dfrac{n-1}{2^n}+\dfrac{n}{2^{n+1}}$ ....(2)

$$\frac{S}{2} = \frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+........+\frac{1}{2^n}-\frac{n}{2^{n+1}} = \frac{\frac{1}{2}\left(1-\frac{1}{2^n}\right)}{1-\frac{1}{2}} - \frac{n}{2^{n+1}} = 1-\frac{1}{2^n}-\frac{n}{2^{n+1}}$$

$$S_n = 2\left[\frac{2^{n+1}-2-n}{2^{n+1}}\right] = \frac{2^{n+1}-2-n}{2^n}. \text{ If } n \to \infty \text{ then } S_\infty = \lim_{n\to\infty}\left[2-\frac{1}{2^{n-1}}-\frac{n}{2^n}\right] = 2$$

Ex.28 If positive square root of, $a^{\frac{1}{a}} . (2a)^{\frac{1}{2a}} . (4a)^{\frac{1}{4a}} . (8a)^{\frac{1}{8a}}$ ..... ∞ is $\dfrac{8}{27}$, find the value of ' a '.

Sol. $a^{\left(\frac{1}{a}+\frac{1}{2a}+\frac{1}{4a}+......\infty\right)} . 2^{\frac{1}{2a}+\frac{2}{4a}+\frac{3}{8a}+......\infty} = \dfrac{8}{27}$

now $\dfrac{1}{a}\left(1+\dfrac{1}{2}+\dfrac{1}{2^2}+........\infty\right) = \dfrac{2}{a}$ and $\dfrac{1}{2a}+\dfrac{2}{4a}+\dfrac{3}{8a}+........\infty = \dfrac{2}{a}$ (use A G P)

$\therefore\ a^{\frac{1}{a}} . 2^{\frac{1}{a}} = \dfrac{8}{27} = \left(\dfrac{1}{3}\right)^3 . 2^3 \Rightarrow a = \dfrac{1}{3}$

Ex.29 Prove the identity $\left(x^{n-1}+\frac{1}{x^{n-1}}\right)+2\left(x^{n-2}+\frac{1}{x^{n-2}}\right)+.....+(n-1)\left(x+\frac{1}{x}\right)+n=\frac{1}{x^{n-1}}\left(\frac{x^n-1}{x-1}\right)^2$.

Sol. The sum on the left may be rewritten as follows

$$\left(\frac{1}{x^{n-1}}+\frac{2}{x^{n-2}}+.....+\frac{n-1}{x}\right)+[x^{n-1}+2x^{n-2}+....+(n-1)x]+n$$

The first bracketed expression is equal to $\frac{1}{x^n}$ $[x+2x^2+....+(n-1)x^{n-1}]=\frac{x[(n-1)x^n-nx^{n-1}+1]}{x^n(x-1)^2}$

The second bracketed expression is obtained from the first one by replacing x by 1/x. Hence, we get the required result.

Ex.30 If $x_1, x_2, ...x_n$ are n non-zero real numbers such that

$(x_1^2+x_2^2+.....+x_{n-1}^2)\ (x_2^2+x_3^2+.....+x_n^2)\le(x_1x_2+x_2x_3+....+x_{n-1}x_n)^2$ then $x_1, x_2, ... x_n$ are in

Sol. We shall make use of the identity $(a_1^2+a_2^2+.....+a_m^2)(b_1^2+b_2^2+.....+b_m^2)-(a_1b_1+a_2b_2+....+a_mb_m)^2$

$=(a_1b_2-a_2b_1)^2+(a_1b_3-a_3b_1)^2+....+(a_{m-1}b_m-a_mb_{m-1})^2$

Thus, $(x_1^2+x_2^2+.....+x_{n-1}^2)(x_2^2+x_3^2+.....+x_n^2)-(x_1x_2+x_2x_3+....+x_{n-1}x_n)^2\le 0$

$\Rightarrow\ (x_1x_3-x_2x_2)^2+(x_1x_4-x_3x_3)^2+.....+(x_{n-2}x_n-x_{n-1}x_{n-1})^2\le 0$

As $x_1, x_2, ...,x_n$ are real, this is possible if and only if

$x_1x_3-x_2^2=x_2x_4-x_3^2=...=x_{n-2}x_n-x^2_{n-1}=0\ \Rightarrow\ \frac{x_1}{x_2}=\frac{x_2}{x_3}=\frac{x_3}{x_4}=....=\frac{x_n}{x_{n-1}}$.

E. HARMONIC PROGRESSION (HP)

A sequence is said to HP if the reciprocals of its terms are in AP .

If the sequence $a_1, a_2, a_3, ... , a_n$ is an HP then $1/a_1, 1/a_2, ... , 1/a_n$ is an AP & converse. Here we do not have the formula for the sum of the n terms of an HP . For HP whose first term is a & second term is b, the n^{th} term is $t_n=\frac{ab}{b+(n-1)(a-b)}$. If a, b, c are in HP $\Rightarrow b=\frac{2ac}{a+c}$ or $\frac{a}{c}=\frac{a-b}{b-c}$.

Ex.31 If a, b, c are in H.P. then prove that $a^3b^3+b^3c^3+c^3a^3=(9ac-6b^2)a^2c^2$.

Sol. $\frac{2}{b}=\frac{1}{a}+\frac{1}{c}\ \Rightarrow\ \frac{1}{a}+\frac{1}{c}-\frac{2}{b}=0$. Use $p+q+r=0\ \Rightarrow\ p^3+q^3+r^3=3pqr$

Ex.32 If $\frac{1}{a}+\frac{1}{c}+\frac{1}{a-b}+\frac{1}{c-b}=0$ prove that a, b, c are in HP unless b = a + c.

Sol. $\Rightarrow$ $\frac{a+c}{ac}+\frac{c-b+a-b}{(a-b)(c-b)} = 0$ $\Rightarrow$ $\frac{a+c}{ac}+\frac{(a+c)-2b}{ac-b(a+c)+b^2} = 0$

Let a + c = λ $\therefore$ $\frac{\lambda}{ac}+\frac{\lambda-2b}{ac-b\lambda+b^2} = 0 \Rightarrow \frac{ac\lambda-b\lambda^2+b^2\lambda+ac\lambda-2abc}{ac(ac-b\lambda+b^2)} = 0$

$\Rightarrow$ $ac\lambda - b\lambda^2 + b^2\lambda + ac\lambda - 2abc = 0$ $\Rightarrow$ $2ac(\lambda - b) - b\lambda(\lambda - b) = 0$ $\Rightarrow$ $(2ac - b\lambda)(\lambda - b) = 0$

$\Rightarrow$ $\lambda = b$ or $\lambda = \frac{2ac}{b}$ $\Rightarrow$ a + c = b or $a + c = \frac{2ac}{b}$ $(\because a + c = \lambda)$

$\Rightarrow$ a + c = b or $b = \frac{2ac}{a+c}$ $\Rightarrow$ a, b, c are in H.P. or a + c = b.

Ex.33 The value of xyz is 55 or $\frac{343}{55}$ according as the series a, x, y, z, b is an A.P. or H.P. Find the values of a & b given that they are positive integers.

Sol. Let a, x, y, z, b are in A.P. $\Rightarrow$ b = a + 4 d $\Rightarrow$ $d = \frac{b-a}{4}$

$\Rightarrow$ $x = a + d = \frac{3a+b}{4}$ $y = a + 2d = \frac{a+b}{2}$ & $z = \frac{a+3b}{4}$

Similarly when a, x, y, z, b are in H.P. then $d = \frac{a-b}{4ab}$

Hence $x = \frac{4ab}{a+3b}$; $y = \frac{2ab}{a+b}$; $z = \frac{4ab}{3a+b}$

In the 1st case $\frac{3a+b}{4} \cdot \frac{a+b}{2} \cdot \frac{a+3b}{4} = 55$ — (1)

In the 2nd case $\frac{4ab}{a+3b} \cdot \frac{2ab}{a+b} \cdot \frac{4ab}{3a+b} = \frac{343}{55}$ — (2)

dividing $a^3 b^3 = 7^3$ $\Rightarrow$ a = 7 ; b = 1 or a = 1 ; b = 7

F. ARITHMETIC, GEOMETRIC & HARMONIC MEANS (AM, GM & HM)

Arithmetic Mean : If three terms are in AP then the middle term is called the AM between the other two, so if a, b, c are in AP, b is AM of a & c.

AM for any n positive number $a_1, a_2, \ldots, a_n$ is ; $A = \frac{a_1+a_2+a_3+\ldots\ldots+a_n}{n}$.

n - Arithmetic Means Between Two Numbers :
If a, b are any two given numbers & a, $A_1, A_2, \ldots, A_n$, b are in AP then $A_1, A_2, \ldots A_n$ are the n AM's between a & b .

$A_1 = a + \frac{b-a}{n+1}$, $A_2 = a + \frac{2(b-a)}{n+1}$, ... , $A_n = a + \frac{n(b-a)}{n+1}$

= a + d , = a + 2d , ... , $A_n = a + nd$, where $d = \frac{b-a}{n+1}$

Note : Sum of n AM's inserted between a & b is equal to n times the single AM between a & b i.e. $\sum_{r=1}^{n} A_r = nA$ where A is the single AM between a & b .

Geometric Means : If a, b, c are in GP, b is the GM between a & c .

$b^2 = ac$, therefore $b = \sqrt{ac}$; $a > 0$, $c > 0$.

n-Geometric Means Between a , b : If a, b are two given numbers & a, G_1, G_2, ... , G_n, b are in GP . Then G_1, G_2, G_3 , ..., G_n are n GMs between a & b.

$G_1 = a(b/a)^{1/n+1}$, $G_2 = a(b/a)^{2/n+1}$, ... , $G_n = a(b/a)^{n/n+1}$
$= ar$, $= ar^2$, ... $= ar^n$, where $r = (b/a)^{1/n+1}$

Note : The product of n GMs between a & b is equal to the nth power of the single GM between a & b i.e. $\prod_{r=1}^{n} G_r = (G)^n$ where G is the single GM between a & b.

Harmonic Mean : If a, b, c are in HP, b is the HM between a & c, then $b = 2ac/[a + c]$.

Ex.34 Prove that the arithmetic mean of the squares of ' n ' quantities in A.P. exceeds the squares of their arithmetic mean by a quantity which depends only upon ' n ' and upon their common difference .

Sol. a + b , a + 2 b , ... , a + n b

Hence $\dfrac{(a+b)^2 + (a+2b)^2 + ... + (a+nb)^2}{n} = \dfrac{1}{n}\left[na^2 + nab(n+1) + b^2\,\dfrac{n(n+1)(2n+1)}{6}\right]$

$= a^2 + ab(n+1) + \dfrac{b^2}{6}(n+1)(2n+1) \quad = a^2 + ab(n+1) + \dfrac{b^2}{6}(2n^2 + 3n + 1)$

Again $\left\{\dfrac{(a+b) + (a+2b) + ... + (a+nb)}{n}\right\}^2 = \left(a + \dfrac{n+1}{2}b\right)^2 = a^2 + ab(n+1) + \dfrac{b^2}{4}(n^2 + 2n + 1)$

Hence A.M. of squares – square of A.M. $= \left(\dfrac{n^2 - 1}{12}\right) b^2$

Ex.35 Two consecutive numbers from 1, 2, 3,...., n are removed. The arithmetic mean of the remaining numbers is $\dfrac{105}{4}$. Find n and those removed numbers.

Sol. Let p and (p + 1) be the removed numbers from 1, 2,...,n then sum of the remaining numbers

$= \dfrac{n(n+1)}{2} - (2p + 1)$

From given condition $\dfrac{105}{4} = \dfrac{\dfrac{n(n+1)}{2} - (2p+1)}{n-2} \quad \Rightarrow \quad 2n^2 - 103n - 8p + 206 = 0$

Since n and p are integers so n must be even let n = 2r. we get $p = \dfrac{4r^2 + 103(1-r)}{4}$

Since p is an integer then (1 – r) must be divisible by 4. Let r = 1 + 4t, we get
$n = 2 + 8t$ and $p = 16t^2 - 95t + 1$, Now $1 \le p < n \Rightarrow 1 \le 16t^2 - 95t + 1 < 8t + 2 \Rightarrow t = 6 \Rightarrow n = 50$ and $p = 7$
Hence removed numbers are 7 and 8.

Ex.36 Between two numbers whose sum is $\frac{13}{6}$, an even number of A.M.'s is inserted, the sum of these means exceeds their number by unity. Find the number of means.

Sol. Let a and b be two numbers and 2n A.M.'s are inserted between a and b then

$\frac{2n}{2}$ (a + b) = 2n + 1 $\Rightarrow$ n $\left(\frac{13}{6}\right)$ = 2n + 1. $\left[\text{given } a+b=\frac{13}{6}\right]$ $\Rightarrow$ n = 6 $\therefore$ Number of means = 12.

Ex.37 Insert 20 A.M. between 2 and 86.

Sol. Here 2 is the first term and 86 is the 22^{nd} term of A.P. so 86 = 2 + (21) d $\Rightarrow$ d = 4
so the series is 2, 6, 10, 14,......., 82, 86q $\therefore$ required means are 6, 10, 14,.....82

Ex.38 A, B and C are distinct positive integers, less than or equal to 10. The arithmetic mean of A and B is 9. The geometric mean of A and C is $6\sqrt{2}$. Find the harmonic mean of B and C.

Sol. A + B = 18 .....(1) AC = 72 .....(2)
There are only two possibilities A = 10 and B = 8 or A = 8 and B = 10
If A = 10 then from (2) C is not an integer. Hence A = 8 and B = 10; C = 9

$\therefore$ H.M. between B and C = $\frac{2\cdot 10\cdot 9}{10+9}$ = $\frac{180}{19}$

Ex.39 Insert 4 H.M. between $\frac{2}{3}$ and $\frac{2}{13}$.

Sol. Let d be the common difference of corresponding A.P. so d = $\frac{\frac{13}{2}-\frac{3}{2}}{5}$ = 1

$\therefore$ $\frac{1}{H_1}=\frac{3}{2}+1=\frac{5}{2}$ $\Rightarrow$ H_1 = $\frac{2}{5}$; $\frac{1}{H_2}=\frac{3}{2}+2=\frac{7}{2}$ $\Rightarrow$ H_2 = $\frac{2}{7}$

$\frac{1}{H_3}=\frac{3}{2}+3=\frac{9}{2}$ $\Rightarrow$ H_3 = $\frac{2}{9}$; $\frac{1}{H_4}=\frac{3}{2}+4=\frac{11}{2}$ $\Rightarrow$ H_4 = $\frac{2}{11}$.

G. SUMMATION OF SERIES

(i) $\sum\limits_{r=1}^{n}$ $(a_r \pm b_r)$ = $\sum\limits_{r=1}^{n}$ a_r ± $\sum\limits_{r=1}^{n}$ b_r . **(ii)** $\sum\limits_{r=1}^{n}$ k a_r = k $\sum\limits_{r=1}^{n}$ a_r .

(iii) $\sum\limits_{r=1}^{n}$ k = nk ; where k is a constant .

Remarks :

(i) $\sum\limits_{r=1}^{n}$ r = $\frac{n\,(n+1)}{2}$ (sum of the first n natural nos.)

(ii) $\sum\limits_{r=1}^{n}$ r^2 = $\frac{n\,(n+1)\,(2n+1)}{6}$ (sum of the squares of the first n natural numbers)

(iii) $\sum\limits_{r=1}^{n} r^3 = \dfrac{n^2(n+1)^2}{4} = \left[\sum\limits_{r=1}^{n} r\right]^2$ (sum of the cubes of the first n natural numbers)

(iv) $\sum\limits_{r=1}^{n} r^4 = \dfrac{n}{30}(n+1)(2n+1)(3n^2+3n-1)$

Method Of Differences :

Let u_1, u_2, u_3...... be a sequence, such that $u_2 - u_1, u_3 - u_2$,............ is either an A.P. or a G.P. then nth term u_n of this sequence is obtained as follows

$S = u_1 + u_2 + u_3 + \ldots\ldots. + u_n$ ...(i) $\quad S = u_1 + u_2 + \ldots\ldots\ldots + u_{n-1} + u_n$ ...(ii)

(i) – (ii) $\Rightarrow$ $u_n = u_1 + (u_2 - u_1) + (u_3 - u_2) + \ldots\ldots\ldots. + (u_n - u_{n-1})$

Where the series $(u_2 - u_1) + (u_3 - u_2) + \ldots\ldots\ldots + (u_n - u_{n-1})$ is

either in A.P. or in G.P. then we can find u_n and hence sum of this series as $S = \sum\limits_{r=1}^{k} u_r$

Remark : It is not always necessary that the series of first order of differences i.e. $u_2-u_1, u_3-u_2 \ldots u_n-u_{n-1}$ is always either in A.P. or in G.P. in such case let $u_1 = T_1$, $u_2 - u_1 = T_2$, $u_3 - u_2 = T_3$ ....,$u_n - u_{n-1} = T_n$.

So $u_n = T_1 + T_2 + \ldots\ldots. + T_n$ .... (i) $\quad u_n = T_1 + T_2 + \ldots\ldots.. + T_{n-1} + T_n$ .... (ii)

(i) – (ii) $\Rightarrow$ $T_n = T_1 + (T_2 - T_1) + (T_3 - T_2) + \ldots\ldots + (T_n + T_{n-1})$

Now, the series $(T_2 - T_1) + (T_3 - T_2) + \ldots.. + (T_n - T_{n-1})$ is series of second order of differences and when it is either in A.P. or in G.P., then $u_n = u_1 + \Sigma T_r$

Otherwise in the similar way we find series of higher order of differences and the n^{th} term of the series. If possible express r^{th} term as difference of two terms as $t_r = f(r) - f(r \pm 1)$. This can be explained with help of examples given below.

Ex.40 Find the sum to n terms of the series,

0.7 + 7.7 + 0.77 + 77.7 + 0.777 + 777.7 + 0.7777 + ... where n is even .

Sol. n = 2m

s = (0.7 + 0.77 + 0.777 + ... m term) + (7.7 + 77.7 + 777.7 + ... m terms)

$= \dfrac{7}{9}(0.9 + 0.99 + 0.999 + \ldots.. \text{ m terms}) + \dfrac{7}{90}((10^2 - 1) + (10^3 - 1) + \ldots.. + (10^{m+1} - 1))$

$= \dfrac{7}{9}\left\{\dfrac{n}{2} - \dfrac{1}{9}\left(1 - \dfrac{1}{10^{n/2}}\right)\right\} + \dfrac{7}{90}\left\{\dfrac{100}{9}\left(10^{n/2} - 1\right) - \dfrac{n}{2}\right\}$

Ex.41 Determine the sums of the following series

1. $1 + \dfrac{3}{2} + \dfrac{5}{4} + \dfrac{7}{8} + \ldots.. + \dfrac{2n-1}{2^{n-1}}$; $\quad$ **2.** $1 - \dfrac{3}{2} + \dfrac{5}{4} - \dfrac{7}{8} + \ldots.. + (-1)^{n-1}\dfrac{2n-1}{2^{n-1}}$.

Sol. To determine the required sums first compute the following sum

$1 + 3x + 5x^2 + \ldots + (2n-1)x^{n-1} = \sum\limits_{k=1}^{n}(2k-1)\, x^{k-1} = 2\sum\limits_{k=1}^{n} kx^{k-1} - \sum\limits_{k=1}^{n} x^{k-1} = \dfrac{2nx^n(x-1) - (x+1)(x^n - 1)}{(x-1)^2}$

For computing the first of the sums put in the deduced formula $x = \dfrac{1}{2}$. then we have

$1 + \dfrac{3}{2} + \dfrac{5}{4} + \dfrac{7}{8} + \ldots. + \dfrac{2n-1}{2^{n-1}} = \dfrac{1}{2^{n-1}}\{3(2^n - 1) - 2n\}$

And putting x = –1/2, we find $1 - \dfrac{3}{2} + \dfrac{5}{4} + \dfrac{7}{8} + \ldots..+(-1)^{n-1}\dfrac{2n-1}{2^{n-1}} = \dfrac{2^n + (-1)^{n+1}(6n+1)}{9.2^{n-1}}$

Ex.42 There are n necklaces such that the first necklace contains 5 beads, the second contains 7 beads and, in general the ith necklace contains i beads more than the number of beads in (i– 1)th necklace. Find the total number of beads is all the n necklaces.

Sol. Let us write the sequence of the number of beads in the 1st, 2nd, 3rd, ..., nth necklaces.

$= 5, 7, 10, 14, 19, \ldots\ldots\ldots = (4 + 1), (4 + 3), (4 + 6)\ (4 + 10), (4 + 15), \ldots\ldots, \left[4 + \frac{n(n+1)}{2}\right]$

S_n = Total number of beads in the n necklaces $S_n = \left\{\underbrace{4 + 4 + \ldots. + 4}_{\text{n times}}\right\} + 1 + 3 + 6 + \ldots. + \frac{n(n+1)}{2}$

$= 4n$ + Sum of the first n triangular numbers $= 4n + \frac{1}{2}\Sigma(n^2 + n) = 4n + \frac{1}{2}(\Sigma n^2 + \Sigma n)$

$= 4n + \frac{1}{2}\left[\frac{n(n+1)(2n+1)}{6}\right] + \frac{1}{2}\frac{n(n+1)}{2} \quad = 4n + \frac{n(n+1)(2n+1)}{12} + \frac{n(n+1)}{4}$

$= \frac{1}{12}[48n + 2n(n + 1)(n + 2)] \qquad = \frac{n}{6}[n^2 + 3n + 26].$

Ex.43 If $S_1, S_2, S_3, \ldots S_n, \ldots$ are the sums of infinite geometric series whose first terms are 1, 2, 3, ... n, ... and whose common ratios are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n+1}, \ldots$ respectively, then find the value of $\sum_{r=1}^{2n-1} S_r^2$.

Sol. r^{th} series will have $a = r$ and common ratio is $\frac{1}{r+1}$

$\therefore \quad S_r = \frac{r}{1 - \frac{1}{r+1}} = \frac{r(r+1)}{r} = r + 1 \quad \therefore \quad S_r^2 = (r + 1)^2$

$\therefore \quad \sum_{r=1}^{2n-1} S_r^2 = \sum_{r=1}^{2n-1}(r+1)^2 = 2^2 + 3^2 + 4^2 + \ldots\ldots + (2n - 1)^2 + (2n)^2 = 1^2 + 2^2 + 3^2 + \ldots. + (2n)^2 - 1$

= sum of the square of the first (2n) natural numbers $= \frac{(2n)(2n+1)(4n+1)}{6} - 1 = \frac{n(2n+1)(4n+1)}{3} - 1$

Ex.44 Find the sum to n-terms 3 + 7 + 13 + 21 + ...

Sol. Let $S = 3 + 7 + 13 + 21 + \ldots\ldots\ldots + T_n$ ....(i)

$S = \quad 3 + 7 + 13 + \ldots\ldots\ldots + T_{n-1} + T_n$ ....(ii)

(i) – (ii) $\Rightarrow$ $T_n = 3 + 4 + 6 + 8 + \ldots\ldots\ldots + (T_n - T_{n-1})$

$= 3 + \frac{n-1}{2}[8 + (n - 2)2] = 3 + (n - 1)(n + 2) = n^2 + n + 1$

Hence $\quad S = \Sigma(n^2 + n + 1) = \Sigma n^2 + \Sigma n + \Sigma 1 = \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} + n = \frac{n}{3}(n^2 + 3n + 5)$

Ex.45 Find the sum of n-terms 1 + 4 + 10 + 22 + ...

Sol. Let $S = 1 + 4 + 10 + 22 + ... + T_n$ .... (i)

$S = 1 + 4 + 10 + ...+T_{n-1} + T_n$ .... (ii)

(i) – (ii) $\Rightarrow$ $T_n = 1 + (3 + 6 + 12 + ... + T_n - T_{n-1}) = 1 + 3\left(\frac{2^{n-1}-1}{2-1}\right) = 3 . 2^{n-1} - 2$

So $S = \Sigma T_n = 3\,\Sigma 2^{n-1} - \Sigma 2 = 3 . \left(\frac{2^n - 1}{2-1}\right) - 2n = 3.2^n - 2n - 3$

Ex.46 Find the nth term and the sum of n term of the series 2, 12, 36, 80, 150, 252

Sol. Let $S = 2 + 12 + 36 + 80 + 150 + 252 + ... + T_n$ .... (i)

$S = 2 + 12 + 36 + 80 + 150 + 252 + ... + T_{n-1} + T_n$ .... (ii)

(i) = (ii) $\Rightarrow T_n = 2 + 10 + 24 + 44 + 70 + 102 + ... + (T_n - T_{n-1})$ .... (iii)

$T_n = 2 + 10 + 24 + 44 + 70 + 102 + ... + (T_{n-1} - T_{n-2}) + (T_n - T_{n-1})$..... (iv)

(iii) – (iv) $\Rightarrow T_n - T_{n-1} = 2 + 8 + 14 + 20 + 26 + ... = \frac{n}{2}\,[4+(n-1)\,6]=n\,[3n-1] = T_n - T_{n-1} = 3n^2 - n$

$\therefore$ general term of given series is $\Sigma\, T_n - T_{n-1} = \Sigma\, 3n^2 - n = n^3 + n^2$.

Hence sum of this sereis is

$S = \Sigma\, n^3 + \Sigma\, n^2 = \frac{n^2(n+1)^2}{4} + \frac{n(n+1)(2n+1)}{6} = \frac{n(n+1)}{12}\,(3n^2 + 7n + 2)\ \frac{1}{12}\, n\,(n + 1)\,(n + 2)\,(3n + 1)$

Ex.47 Find the general term and sum of n terms of the series 9, 16, 29, 54, 103

Sol. Let $S = 9 + 16 + 29 + 54 + 103 + ... + T_n$ .... (i)

$S = 9 + 16 + 29 + 54 + 103 + ...+ T_{n-1} + T_n$ .... (ii)

(i) – (ii) $\Rightarrow T_n = 9 + 7 + 13 + 25 + 49 + ... + (T_n - T_{n-1})$ .... (iii)

$T_n = 9 + 7 + 13 + 25 + 49 + ... + (T_{n-1} - T_{n-2}) + (T_n - T_{n-1})$ .... (iv)

(iii) – (iv) $\Rightarrow T_n - T_{n-1} = 9 + (-2) + \underbrace{6+12+24+......}_{(n-2)\text{terms}} = 7 + 6\,[2^{n-2} - 1] = 6(2)^{n-2} + 1$.

$\therefore$ General term is $T_n = 6(2)^{n-1} + n + 2$

Also sum $S = \Sigma T_n = 6\Sigma 2^{n-1} + \Sigma n + \Sigma 2 = 6.\,\frac{(2^n - 1)}{2-1} + \frac{n(n+1)}{2} + 2n + 2n = 6(2^n - 1) + \frac{n(n+5)}{2}$

Ex.48 The n^{th} term, a_n of a sequence of numbers is given by the formula $a_n = a_{n-1} + 2n$ for $n \geq 2$ and $a_1 = 1$. Find an equation expressing a_n as a polynomial in n. Also find the sum to n terms of the sequence.

Sol. $a_1 = 1$; $a_2 = a_1 + 4 = 1 + 4 = 5$; $a_3 = a_2 + 6 = 5 + 6 = 11$; $a_4 = a_3 + 8 = 11 + 8 = 19$ and so on

hence $S = 1 + 5 + 11 + 19 + ... + a_n$

$S = \quad + 1 + 5 + 11 + ... + a_{n-1} + a_n$

(–) ——————————————————

$0 = 1 + 4 + 6 + 8 + ... + (a_n - a_{n-1}) - a_n$

$a_n = 1 + 2\underbrace{\left(2+3+4+.......+(a_n - a_{n-1})\right)}_{(n-1)\text{ terms}}$

$a_n = 1 + 2\cdot\frac{n-1}{2}\,[4 + (n - 2)] = 1 + (n - 1)\,(n + 2) = n^2 + n - 1$

sum $= \sum a_n = \sum n^2 + \sum n - \sum 1 = \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} - n\ \frac{n}{6}[(n + 1)(2n + 1) + 3(n + 1) - 6]$

$= \frac{n}{6}[2n^2 + 3n + 1 + 3n + 3 - 6] = \frac{n}{6}[2n^2 + 6n - 2] = \frac{n(n^2 + 3n - 1)}{3}$

Ex.49 Find the n^{th} term of the series 1 + 2 + 5 + 12 + 25 + 46 + ...

Sol. Let the sum of the series by S_n and nth term of the series be T_n

Then $S_n = 1 + 2 + 5 + 12 + 46 + ...+T_{n-1} + T_n$ ...(1)

$\therefore$ $S_n = \quad 1 + 2 + 5 + 12 + 25 + ... + T_{n-1} + T_n$ ...(2)

Subtracting (2) from (1), we get $0 = 1 + 1 + 3 + 7 + 13 + 21 + ... + (T_n - T_{n-1}) - T_n$

$\therefore$ $T_n = 1 + 3 + 7 + 13 + 21 + ...+t_{n-1} + t_n$ ...(3)

(Here nth term of T_n is t_n)

$\therefore$ $T_n = 1 + 1 + 3 + 7 + 13 + ... \quad t_{n-1} + t_n$ ...(4)

Subtracting (4) from (3), we get $0 = 1 + 0 + 2 + 4 + 6 + 8 + ... +(t_n - t_{n-1}) - t_n$

$\therefore$ $t_n = 1 + 2 + 4 + 6 + 8 +....(n - 1)$ terms $= 1 + (2 + 4 + 6 + 8\ (n - 2)$ terms)

$$= 1 + \frac{(n-2)}{2}\{2.2 + (n - 2 - 1)\,2\} = 1 + (n - 2)(2 + n - 3) \quad \therefore \quad t_n = n^2 - 3n + 3$$

then $\quad T_n = \Sigma t_n = \Sigma n^2 - 3\,\Sigma n + 3\,\Sigma 1 \quad = \dfrac{n(n+1)(2n+1)}{6} - \dfrac{3n(n+1)}{2} + \dfrac{3n}{1}$

$$= \frac{n}{6}\{2n^2 + 3n + 1 - 9n - 9 + 18\} \quad \therefore \quad T_n = \frac{n}{6}(2n^2 - 6n + 10) = \frac{n}{3}(n^2 - 3n + 5)$$

Alternative Method : The nth terms of the series can be written directly from the following procedure

1, 2, 5, 12, 25, 46,...... (given series)
1, 3, 7, 13, 21,..... (first consecutive differences)
2, 4, 6, 8,.... (second consecutive differences)
2, 2, 2,.... (constant terms)

then $\quad T_n = a(n - 1)(n - 2)(n - 3) + b(n - 1)(n - 2) + c(n - 1) + d$

Putting $\quad n = 1, 2, 3, 4$ then we get $T_4 = 6a + 6b + 3c + d = 12$

$\therefore$ $a = \dfrac{1}{3}$, $b = 1$, $c = 1$, $d = 1$ $\quad$ Hence $\quad T_n = \dfrac{n}{3}(n^2 - 3n + 5)$

Ex.50 The squares of the natural numbers are grouped like (1^2); $(2^2, 3^2, 4^2)$; $(5^2, 6^2, 7^2, 8^2, 9^2)$; and so on. Find the sum of the elements in n^{th} group.

Sol. By observations, the last element of the n^{th} group $= (n^2)^2$

The number of elements in n^{th} group $= (2n - 1)$

The first element of the n^{th} group $= (n^2 - 2n + 2)^2$

Hence sum of the numbers in n^{th} group are

$S = (n^2 - 2n + 2)^2 + (n^2 - 2n + 3)^2 + ... + (n^2)^2$

$\Rightarrow$ $S = [1^2 + 2^2 + 3^2 + ...\ (n^2)^2] - [1^2 + 2^2 + 3^2 + ((n - 1)^2)^2]$

$$\Rightarrow \quad S = \frac{n^2(n^2+1)(2n^2+1)}{6} - \frac{(n-1)^2(n^2-2n+2)(2n^2-4n+3)}{6}$$

Ex.51 The natural numbers are arranged in groups as given below ;

				1				
			2		3			
		4	5		6	7		
8	9	10	11		12	13	14	15

....

Prove that the sum of the numbers in the n^{th} group is $2^{n-2}\{2^n + 2^{n-1} - 1\}$.

Sol. Note that n^{th} group has 2^{n-1} terms . 1^{st} term in the n^{th} group is 2^{n-1} and the last term in the n^{th} group is $2^n - 1$. $\therefore$ sum of the terms in the n^{th} group $= \dfrac{N}{2}(A + L)$; $N = 2^{n-1}$; $A = 2^{n-1}$; $L = 2^n - 1$

Ex.52 Find the n^{th} term and the sum of n terms of the series 1, 2, 3, 6 17, 54, 171.....

Sol. To form the successive order of differences

1 2 3 6 17 54 171 ... ...

1 1 3 11 37 117 ... ...

0 2 8 26 80 ... ...

2 6 18 54 ... ...

The terms of the third order of differences are in G.P. with the common ratio 3. For this we may assume that $u_n = a3^{n-1} + bn^2 + cn + d$. To determine the constants a, b, c, d, we make n equal to 1, 2, 3, 4 successively. Then $a + b + c + d = 1, 3a + 4b + 2c + d = 2,$

$9a + 9b + 3c + d = 3,$ $27a + a6b + 4c + d = 6.$

From these equations we have a = $\frac{1}{4}3^{n-1} - \frac{1}{2}n^2 + 2n - \frac{3}{4}$.

Now $S_n = \sum_{n=1}^{n} u_n = \frac{1}{4}.\frac{3^n - 1}{3-1} - \frac{1}{2}.\frac{n(n+1)(2n+1)}{6} + 2\frac{n(n+1)}{2} - \frac{3n}{4} = \frac{1}{8}(3^n - 1) - \frac{n}{12}(2n^2 - 9n - 2).$

Ex.53 Find the sum to n-terms of the series 1.2 + 2.3 + 3.4 + ...

Sol. Let T_r be the general term of the series. So $T_r = r(r + 1)$

To express $t_r = f(r) - f(r - 1)$ multiply and divide t_r by $[(r + 2) - (r - 1)]$

So $T_r = \frac{r}{3}(r + 1)[(r + 2) - (r - 1)] = \frac{1}{3}[r(r + 1)(r + 2) - (r - 1)r(r + 1)].$

Let $f(r) = \frac{1}{3}r(r + 1)(r + 2)$ so $T_r = [f(r) - f(r - 1)].$

Now $S = \sum_{r=1}^{n} T_r = T_1 + T_2 + T_3 + ... + T_n$

$T_1 = \frac{1}{3}[1 . 2 . 3 - 0]$

$T_2 = \frac{1}{3}[2 . 3 . 4 - 1 . 2 . 3]$

$T_3 = \frac{1}{3}[3 . 4 . 5 - 2 . 3 . 4]$

$\vdots$

$T_n = \frac{1}{3}[n(n + 1)(n + 2) - (n - 1)n(n + 1)]$

$\therefore$ $S = \frac{1}{3}n(n + 1)(n + 2)$. Hence sum of series is $f(n) - f(0)$.

Ex.54 Sum to n terms of the series $\frac{1}{(1+x)(1+2x)}+\frac{1}{(1+2x)(1+3x)}+\frac{1}{(1+3x)(1+4x)}+......$

Sol. Let T_r be the general term of the series $T_r = \frac{1}{(1+rx)(1+(r+1)x)}$

So $T_r = \frac{1}{x}\left[\frac{[1+(r+1)x]-(1+rx)}{(1+rx)(1+(r+1)x)}\right] = \frac{1}{x}\left[\frac{1}{1+rx}-\frac{1}{1+(r+1)x}\right]$

$T_r = f(r) - f(r + 1)$

$\therefore \quad S = \Sigma T_r = T_1 + T_2 + T_3 + ... + T_n = \frac{1}{x}\left[\frac{1}{1+rx}-\frac{1}{1+(n+1)x}\right] = \frac{n}{(1+x)[1+(n+1)x]}$

Ex.55 **(a)** Sum the following series to infinity $\frac{1}{1\cdot4\cdot7} + \frac{1}{4\cdot7\cdot10} + \frac{1}{7\cdot10\cdot13} + ...$

(b) Sum the following series upto n-terms $1 \cdot 2 \cdot 3 \cdot 4 + 2 \cdot 3 \cdot 4 \cdot 5 + 3 \cdot 4 \cdot 5 \cdot 6 + ...$

Sol. **(a)** $T_n = \frac{1}{[1+(n-1)3][1+3n][4+3n]} = \frac{1}{(3n-2)(2n+1)(3n+4)} = \frac{1}{6}\left[\frac{1}{(3n-2)(3n+1)}-\frac{1}{(3n+1)(3n+4)}\right]$

$T_1 = \frac{1}{6}\left[\frac{1}{1\cdot4}-\frac{1}{4\cdot7}\right]$

$T_2 = \frac{1}{6}\left[\frac{1}{4\cdot7}-\frac{1}{7\cdot10}\right]$

$\vdots \quad \vdots \quad \vdots$

$T_n = \frac{1}{6}\left[\frac{1}{(3n-2)(3n+1)}-\frac{1}{(3n+1)(3n+4)}\right]$

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$\therefore \quad S_n = \sum T_n = \frac{1}{6}\left[\frac{1}{1\cdot4}-\frac{1}{(3n+1)(3n+4)}\right] = \left[\frac{1}{24}-\frac{1}{6(3n+1)(3n+4)}\right]$ as $n \to \infty$ $\therefore S_\infty = \frac{1}{24}$

(b) $1 \cdot 2 \cdot 3 \cdot 4 + 2 \cdot 3 \cdot 4 \cdot 5 + 3 \cdot 4 \cdot 5 \cdot 6 + ...$

$T_n = \frac{1}{5}n(n + 1)(n + 2)(n + 3) [(n + 4) - (n - 1)]$

$\therefore \quad T_1 = \frac{1}{5}[1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 - 0]$

$T_2 = \frac{1}{5}[2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 - 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5]$

$\vdots \quad \vdots \quad \vdots$

$T_n = \frac{1}{5}[n(n + 1)(n + 2)(n + 3)(n + 4) - (n - 1)n(n + 1)(n + 2)(n + 3)]$

$\therefore \quad S_n = \sum T_n = \frac{1}{5}[n(n + 1)(n + 2)(n + 3)(n + 4)] = \frac{1}{5}n(n + 1)(n + 2)(n + 3)(n + 4)$

Ex.56 Find the sum to n terms and if possible also to infinity $\frac{1}{1.3.7} + \frac{1}{3.5.9} + \frac{1}{5.7.11} + ...$

Sol. $\frac{5}{1.3.5.7} + \frac{7}{3.5.7.9} + \frac{9}{5.7.9.11} + ...\ \infty$

$$T_n = \frac{(2n+3)}{(2n-1)\,(2n+1)\,(2n+3)\,(2n+5)} = \frac{(2n+5)-2}{(2n-1)\,(2n+1)\,(2n+3)\,(2n+5)}$$

$$= \frac{1}{(2n-1)\,(2n+1)\,(2n+3)} - \frac{2}{(2n-1)\,(2n+1)\,(2n+3)\,(2n+5)}$$

$$= \frac{1}{4}\left[\frac{1}{(2n-1)\,(2n+1)} - \frac{1}{(2n+1)\,(2n+3)}\right]$$

$$- \frac{2}{6}\left[\frac{1}{(2n-1)\,(2n+1)\,(2n+3)} - \frac{1}{(2n+1)\,(2n+3)\,(2n+5)}\right]$$

$$T_1 = \frac{1}{4}\left[\frac{1}{1.3} - \frac{1}{3.5}\right] - \frac{1}{3}\left[\frac{1}{1.3.5} - \frac{1}{3.5.7}\right]$$

$$T_2 = \frac{1}{4}\left[\frac{1}{3.5} - \frac{1}{5.7}\right] - \frac{1}{3}\left[\frac{1}{3.5.7} - \frac{1}{5.7.9}\right]$$

$$\therefore\ S_n = \frac{1}{4}\left[\frac{1}{1.3} - \frac{1}{(2n+1)\,(2n+3)}\right] - \frac{1}{3}\left[\frac{1}{1.3.5} - \frac{1}{(2n+1)\,(2n+3)\,(2n+5)}\right]$$

$$S_\infty = \frac{1}{1.3}\left[\frac{1}{4} - \frac{1}{15}\right] = \frac{11}{60}\,.\,\frac{1}{3} = \frac{11}{180}$$ and

$$S_n = \frac{11}{180} - \frac{1}{(2n+1)\,(2n+3)}\left[\frac{1}{4} - \frac{1}{3\,(2n+5)}\right] = \frac{11}{180} - \frac{6n+1}{12\,(2n+1)\,(2n+3)\,(2n+5)}$$

Ex.57 Evaluate the sum , $\sum_{k=1}^{n} \frac{1}{k\,(k+1)\,(k+2)\,......\,(k+r)}$.

Sol. $S = \frac{1}{1.2.3\,.....\,(r+1)} + \frac{1}{2.3.4\,.....\,(r+2)} + ... + \frac{1}{n\,(n+1)\,(n+2)\,.....\,(n+r)}$

$$T_1 = \frac{1}{r}\left[\frac{1}{1.2.3\,......\,r} - \frac{1}{2.3.4\,......\,(r+1)}\right]$$

$$T_2 = \frac{1}{r}\left[\frac{1}{2.3.4\,......\,(r+1)} - \frac{1}{3.4.5\,......\,(r+2)}\right]$$

..

$$T_n = \frac{1}{r}\left[\frac{1}{n\,(n+1)\,(n+2)\,......\,(n+r)-1} - \frac{1}{(n+1)\,(n+2)\,(n+3)\,......\,(n+r)}\right] \Rightarrow \text{Sum} = \frac{1}{r}\left[\frac{1}{r\,!} - \frac{n\,!}{(n+r)\,!}\right]$$

Ex.58 Sun to n terms of the series $\frac{4}{1.2.3}+\frac{5}{2.3.4}+\frac{6}{3.4.5}$ + ...

Sol. Let $T_r = \frac{r+3}{r(r+1)(r+2)} = \frac{1}{(r+1)(r+2)}+\frac{3}{r(r+1)(r+2)} = \left[\frac{1}{r+1}-\frac{1}{r+2}\right]+\frac{3}{2}\left[\frac{1}{r(r+1)}-\frac{1}{(r+1)(r+2)}\right]$

$\therefore$ $S=\left[\frac{1}{2}-\frac{1}{n+2}\right]+\frac{3}{2}\left[\frac{1}{2}-\frac{1}{(n+1)(n+2)}\right]=\frac{5}{4}-\frac{1}{n+2}\left[1+\frac{3}{2(n+1)}\right]=\frac{5}{4}-\frac{1}{2(n+1)(n+2)}[2n+5]$

Ex.59 Sum of the series $\frac{n}{1.2.3}+\frac{n-1}{2.3.4}+\frac{n-2}{3.4.5}+ \ldots + \frac{1}{n(n+1)(n+2)}$ is $\frac{n^2+3}{4(n+1)}$

Sol. t_r, the *r*th term of the series is given by

$$t_r = \frac{n-r+1}{r(r+1)(r+2)} = (n+1)\frac{1}{r(r+1)(r+2)} - \frac{1}{(r+1)(r+2)}$$

$$= (n+1)\left[\frac{1}{2r}-\frac{1}{r+1}+\frac{1}{2(r+2)}\right]-\left[\frac{1}{r+1}-\frac{1}{r+2}\right]$$ [resolve into partial fractions]

$$= (n+1)\left[\frac{1}{2}\left(\frac{1}{r}-\frac{1}{r+1}\right)-\frac{1}{2}\left(\frac{1}{r+1}-\frac{1}{r+2}\right)\right]-\left[\frac{1}{r+1}-\frac{1}{r+2}\right] = \frac{n+1}{2}\left(\frac{1}{r}-\frac{1}{r+1}\right)-\left(\frac{n+1}{2}+1\right)\left(\frac{1}{r+1}-\frac{1}{r+2}\right)$$

$$\Rightarrow \sum_{r=1}^{n} t_r = \left(\frac{n+1}{2}\right)\left(1-\frac{1}{n+1}\right) - \left(\frac{n+3}{2}\right)\left(\frac{1}{2}-\frac{1}{n+2}\right) = \frac{n}{2}-\frac{n+3}{4}+\frac{n+3}{2(n+2)}=\frac{n^2+3}{4(n+2)}$$

Ex.60 Find the sum of n terms of the series $\frac{1}{2.3}.2+\frac{2}{3.4}.2^2+\frac{3}{4.5}.2^3+\ldots$

Sol. Here $u_n = \frac{n}{(n+1)(n+2)}.2^n$. Let $\frac{n}{(n+1)(n+2)}=\frac{A}{n+1}+\frac{B}{n+2}$, or, $n = A(n+2) + B(n+1)$.

Equating the coeffs. of like powers of n, $A + B = 1$, $2A + B = 0$. $\therefore$ $A = -1, B = 2$.

Now we may write $u_n = \frac{2^{n+1}}{n+2}-\frac{2^n}{n+1}$. Putting n = 1, 2, 3,...., n, we have

$u_1 = \frac{2^2}{3}-\frac{2}{2}$, $u_2 = \frac{2^3}{4}-\frac{2^2}{3}$, $u_3 = \frac{2^4}{5}-\frac{2^3}{4}$, ... $u_n = \frac{2^{n+1}}{n+2}-\frac{2^n}{n+1}$. By addition, $S_n = \frac{2^{n+1}}{n+2} - 1$.

Ex.61 Find the sum of the infinite series $\frac{3}{1^2.2^2}+\frac{5}{2^2.3^2}+\frac{7}{3^2.4^2}+\ldots..$

Sol. Here $u_n = \frac{2n+1}{n^2(n+1)^2} = \frac{(n+1)^2-n^2}{n^2(n+1)^2} = \frac{1}{n^2} - \frac{1}{(n+1)^2}$

$\therefore$ $u_1 = \frac{1}{1^2}-\frac{1}{2^2}$, $u_2 = \frac{1}{2^2} - \frac{1}{3^2}$ .... $u_n = \frac{1}{n^2} - \frac{1}{(n+1)^2}$ By addition, $S_n = 1 - \frac{1}{(n+1)^2}$

If $n \to \infty$, then $S_n \to 1$. Hence the sum of the infinite series is 1.

Ex.62 Sum to n terms of the series $\frac{1}{x+1}+\frac{2x}{(x+1)(x+2)}+\frac{3x^2}{(x+1)(x+2)(x+3)}+...$

Sol. $u_n = \frac{nx^{n-1}}{(x+1)(x+2)....(x+n)} = \frac{(x+n)x^{n-1}-x^n}{(x+1)(x+2)....(x+n)} = \frac{x^{n-1}}{(x+1)(x+2)....(x+n-1)} - \frac{x^n}{(x+1)(x+2)....(x+n)}$

$u_1 = 1 - \frac{x}{x+1}$, $u_2 = \frac{x}{x+1}-\frac{x^2}{(x+1)(x+2)}$, $u_3 = \frac{x^2}{(x+1)(x+2)}-\frac{x^3}{(x+1)(x+2)(x+3)}$

.. ..

$u_n = \frac{x^{n-1}}{(x+1)(x+2)...(x+n-1)}-\frac{x^n}{(x+1)(x+2)....(x+n)}$ $\therefore S_n = 1 - \frac{x^n}{(x+1)(x+2)....(x+n)}$.

Ex.63 Let $s_n = 1 + \frac{1}{2}+\frac{1}{3}+......+\frac{1}{n}$. Show that

(A) $s_n = n - \left(\frac{1}{2}+\frac{2}{3}+......+\frac{n-1}{n}\right)$; **(B)** $ns_n = n + \left(\frac{n-1}{1}+\frac{n-2}{2}+.......+\frac{2}{n-2}+\frac{1}{n-1}\right)$.

Sol. **(A)** It is obvious that

$s_n = 1 + \frac{1}{2}+\frac{1}{3}+....+\frac{1}{n} = n + \left[(1-1)+\left(\frac{1}{2}-1\right)+\left(\frac{1}{3}-1\right)+.......+\left(\frac{1}{n}-1\right)\right] = n - \left(\frac{1}{2}+\frac{2}{3}+.....+\frac{n-1}{n}\right)$.

(B) $s_n = \sum_{k=1}^{k=n}\frac{1}{k}$, $ns_n = \sum_{k=1}^{k=n}\frac{n-k+k}{k}\sum_{k=1}^{k=n}\left(\frac{n-k}{k}+1\right)$. Hence, $s_n = n + \left(\frac{n-1}{1}+\frac{n-2}{2}+.....+\frac{1}{n-1}\right)$.

Ex.64 If $f(t) = 1 + \frac{1}{2}+\frac{1}{3}+......+\frac{1}{t}$, prove that $\sum_{k=1}^{n}(2k+1)\ f(k) = (n+1)^2 \cdot f(n) - \frac{n(n+1)}{2}$

Sol. L H S = $3f(1) + 5f(2) + 7f(3) + ... + (2n+1)\ f(n)$

$= 3 + 5\left(1+\frac{1}{2}\right) + 7\left(1+\frac{1}{2}+\frac{1}{3}\right) + ... + (2n+1)\left[1+\frac{1}{2}+\frac{1}{3}+......+\frac{1}{n}\right]$

$= \frac{1}{1}[3 + 5 + 7 + ... + (2n+1)] + \frac{1}{2}[5 + 7 + ... + (2n+1)]$

$+ \frac{1}{3}[7 + 9 + ... + (2n+1)] + ... + \frac{1}{n}(2n+1)$

$= [(n+1)^2 - 1] + \frac{1}{2}[(n+1)^2 - 2^2] + \frac{1}{3}[(n+1)^2 - 3^2] + ... + \frac{1}{n}[(n+1)^2 - n^2]$

$= (n+1)^2\left[1+\frac{1}{2}+\frac{1}{3}+.......+\frac{1}{n}\right] - (1+2+3+......+n) = (n+1)^2\ f(n) - \frac{n(n+1)}{2}$

Ex.65 If $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ... = \log_e 2$, then sum $\frac{1}{1.3} + \frac{1}{2.5} + \frac{1}{3.7} + \frac{1}{4.9} + ...$

Sol. If S denote the sum of the given series, then

$$S = \frac{2}{2.3} + \frac{2}{4.5} + \frac{2}{6.7} + \frac{2}{8.9} + ... = 2\left[\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\left(\frac{1}{6}-\frac{1}{7}\right)+\left(\frac{1}{8}-\frac{1}{9}\right)+.....\right]$$

$$= 2\left[1-\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-...\right)\right] = 2\,[1 - \log_e 2] = 2 - 2\log_e 2.$$

Ex.66 Find the value of the expression $\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j} 1$

Sol. $\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j} 1 = \sum_{i=1}^{n}\sum_{j=1}^{i} j = \sum_{i=1}^{n}\frac{i(i+1)}{2} = \frac{1}{2}\left[\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i\right]$

$$\frac{1}{2}\left[\frac{n(n+1)(2n+1)}{6}+\frac{n(n+1)}{2}\right] = \frac{n(n+1)}{12}\,[2n + 1 + 3] = \frac{n(n+1)(n+2)}{6}$$

H. A.M. ≥ G.M. ≥ H.M. INEQUALITIES

Theorem : If A, G, H are respectively AM, GM, HM between a & b both being unequal & positive then,
(i) $G^2 = AH$ **(ii)** $A > G > H$ $(G > 0)$. Note that A, G, H constitute a GP.

Let $a_1, a_2, a_3, ... a_n$ be n positive real numbers, then we define their

$$\text{A.M.} = \frac{a_1 + a_2 + a_3 + ... + a_n}{n}, \ \text{G.M.} = (a_1a_2a_3..........a_n)^{1/n} \text{ and H.M.} = \frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+.....+\frac{1}{a_n}}$$

It can be shown that A.M. ≥ G.M. ≥ H.M. and equality holds at either places iff
$a_1 = a_2 = a_3 = ... = a_n$

Ex.67 If a, b, c, d be four distinct positive quantities in G.P., then show that

(a) $a + d > b + c$ **(b)** $\frac{1}{cd}+\frac{1}{ab} > 2\left(\frac{1}{bd}+\frac{1}{ac}-\frac{1}{ad}\right)$.

Sol. Since a, b, c d are in G.P., therefore

(a) using A.M. > G.M., we have for the first three terms $\frac{a+c}{2} > b$ i.e. $a + c > 2b$ ....(1)

$\frac{b+d}{2} > c$ i.e. $b + d > 2c$ ....(2)

From results (1) and (2), we have $a + c + b + d > 2b + 2c$ i.e. $a + d > b + c$
which is the desired result.

(b) using G.M. > H.M., we have for the first three terms $b > \frac{2ac}{a+c}$ i.e. $ab + bc > 2ac$ ....(3)

and for the last three terms $c > \frac{2bd}{b+d}$ i.e. $bc + cd > 2bd$ ....(4)

From results (3) and (4), we have $ab + cd + 2bc > 2ac + 2bd$ i.e. $ab + cd > 2(ac + bd - bc)$

i.e. $\frac{1}{cd}+\frac{1}{ab} > 2\left(\frac{1}{bd}+\frac{1}{ac}-\frac{1}{ad}\right)$ [dividing both sides by abcd] which is the desired result.

Ex.68 If a, b, c are in H.P. and they are distinct and positive then prove that $a^n + c^n > 2b^n$

Sol. Let a^n and c^n be two numbers then $\frac{a^n+c^n}{2} > (a^n c^n)^{1/2} \Rightarrow a^n + c^n > 2(ac)^{n/2}$ .....(i)

Also G.M. > H.M. i.e. $\sqrt{ac} > b$ $(ac)^{n/2} > b^n$.....(ii) hence from (i) and (ii) $a^n + c^n > 2b^n$

Ex.69 Show that $n^n\left(\frac{n+1}{2}\right)^{2n} > (n\,!)^3$.

Sol. Consider the unequal positive numbers $1^3, 2^3, ...n^3$.

Since A.M. > G.M., therefore $\frac{1^3+2^3+......+n^3}{n} > (1^3 . 2^3 ....n^3)^{1/n}$, i.e., $\frac{n(n+1)^2}{4} > \{(n\,!)^3\}^{1/n}$.

Raising both sides to power n, we have $n^n\left(\frac{n+1}{2}\right)^{2n} > (n\,!)^3$.

Ex.70 If a, b, c are positive real numbers such that a + b + c = 1, then find the minimum value of $\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ac}$.

Sol. $\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ac} = \frac{a+b+c}{abc} = \frac{1}{abc}$

Also, $\frac{a+b+c}{3} \geq (abc)^{1/3} \Rightarrow abc \leq \frac{1}{27} \Rightarrow \frac{1}{abc} \geq 27$. Hence $\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ac} \geq 27$

Ex.71 In the equation $x^4 + px^3 + qx^2 + rx + 5 = 0$ has four positive real roots, then find the minimum value of pr.

Sol. Let $\alpha, \beta, \gamma, \delta$ be the four positive real roots of the given equation. Then

$\alpha+\beta+\gamma+\delta = -p$, $\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta = q$, $\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta = -r$, $\alpha\beta\gamma\delta = 5$.

Using A.M. ≥ G.M. $\Rightarrow$ $\frac{\alpha+\beta+\gamma+\delta}{4}.\frac{\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta}{4} \geq \sqrt[4]{\alpha\beta\gamma\delta}\,\sqrt[4]{\alpha^3\beta^3\gamma^3\delta^3} = \alpha\beta\gamma\delta = 5$

$\Rightarrow \left(-\frac{p}{4}\right)\left(-\frac{r}{4}\right) \geq 5 \Rightarrow pr \geq 80 \Rightarrow$ minimum value of pr = 80.

Ex.72 If S = a+b+ c then prove that $\frac{S}{S-a}+\frac{S}{S-b}+\frac{S}{S-c}>\frac{9}{2}$ where a , b & c are distinct positive reals.

Sol. $\frac{(S-a)+S-b)+(S-c)}{3} > \left[(S-a)\ (S-b)\ (S-c)\right]^{1/3}$ ....(i)

& $\frac{1}{3}\left[\frac{1}{S-a}+\frac{1}{S-b}+\frac{1}{S-c}\right] > \left[\frac{1}{S-a}+\frac{1}{S-b}+\frac{1}{S-c}\right]^{1/3}$ ....(ii)

Multiple (i) & (ii) $= \frac{1}{9}\ 2\,(a+b+c)\left[\frac{1}{S-a}+\frac{1}{S-b}+\frac{1}{S-c}\right] > 1$

$= \frac{2}{9}\left[\frac{S}{S-a}+\frac{S}{S-b}+\frac{S}{S-c}\right] > 1$ or $\frac{S}{S-a}+\frac{S}{S-b}+\frac{S}{S-c}>\frac{9}{2}$

Ex.73 If $a_1, a_2, \ldots\ldots, a_n$ are all positive, then show that $\sqrt{a_1a_2}+\sqrt{a_1a_3}+\ldots.+\sqrt{a_{n-1}a_n} \le \frac{(n-1)}{2}(a_1+a_2+\ldots..+a_n)$.

Sol. Add the n (n –1) /2 inequalities $\sqrt{a_1a_2}\le\frac{a_1+a_2}{2}$, $\sqrt{a_1a_3}\le\frac{a_1+a_3}{2}$, ..., $\sqrt{a_{n-1}a_n}\le\frac{a_{n-1}+a_n}{2}$

and note that in the sum on the right each a_i occurs n – 1 times.

For example, for n = 4, we have to consider 6 terms : $\sqrt{(a_1a_2)}$, $\sqrt{(a_1a_3)}$, $\sqrt{(a_1a_4)}$, $\sqrt{(a_2a_3)}$, $\sqrt{(a_2a_4)}$, $\sqrt{(a_3a_4)}$.

Ex.74 Let a > 1 and $n \in N$. Prove the inequality, $a^n - 1 \ge n\left(a^{\frac{n+1}{2}} - a^{\frac{n-1}{2}}\right)$.

Sol. Put $a = \alpha^2$

T P T $\Rightarrow \alpha^{2n}-1 \ge n\,(\alpha^{n+1}-\alpha^{n-1}) \Rightarrow \alpha^{2n}-1 \ge n\,\alpha^{n-1}(\alpha^2-1) \Rightarrow \frac{\alpha^{2n}-1}{\alpha^2-1} \ge n\,.\,\alpha^{n-1}$

$\Rightarrow 1+\alpha^2+\alpha^4+\ldots\ldots+a^{2(n-1)} \ge n\ \alpha^{n-1} \ge \sqrt[n]{1\,.\,\alpha^2\,.\,\alpha^4\,.\ldots\ldots.\,\alpha^{2(n-1)}}$. Which is True as A.M. ≥ G.M.

Ex.75 If s be the sum of n positive unequal quantities a, b, c then prove the inequality ;

$\frac{s}{s-a}+\frac{s}{s-b}+\frac{s}{s-c}+\ldots\ldots > \frac{n^2}{n-1}$ (n ≥ 2).

Sol. AM > GM $\Rightarrow$ $[(s-a)+(s-b)+(s-c)+\ldots\ldots] > n\,[(s-a)+(s-b)+(s-c)+\ldots\ldots]^{1/n}$

$$\Rightarrow \left[\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}+\ldots\ldots\right] > n\left[\frac{1}{(s-a)\ (s-b)\ (s-c)}\right]^{1/n}$$

Multiplying $[(s-a)+(s-b)+\ldots\ldots]\left[\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}+\ldots\ldots\right] > n^2$

or $\left[\frac{(s-a)}{s}+\frac{(s-b)}{s}+\frac{(s-c)}{s}+\ldots\ldots\right]\left[\frac{s}{s-a}+\frac{s}{s-b}+\frac{s}{s-c}+\ldots\ldots\right] > n^2$

$$\Rightarrow \left(\frac{s}{s-a}+\frac{s}{s-b}+\frac{s}{s-c}+\ldots\ldots\right)\left(\frac{ns-s}{s}\right) > n^2 \quad \text{or} \quad \frac{s-a}{s}+\frac{s-b}{s}+\frac{s-c}{s}+\ldots\ldots > \frac{n^2}{n-1}$$

Ex.76 If a, b, c, d are all positive and the sum of any three is greater than twice the fourth, then show that, $abcd > (b+c+d-2a)\ (c+d+a-2b)\ (d+a+b-2c)\ (a+b+c-2d)$.

Sol. Use A.M. > G.M.

$a+b+c-2d = m_1,\ b+c+d-2a = m_2,\ c+d+a-2b = m_3,\ d+a+b-2c = m_4$

Now $m_1+m_2+m_3 = 3c \Rightarrow c = \frac{m_1+m_2+m_3}{3} > (m_1\,m_2\,m_3)^{1/3}$

$m_2+m_3+m_4 = 3d \Rightarrow d = \frac{m_2+m_3+m_4}{3} > (m_2\,m_3\,m_4)^{1/3}$

$m_3+m_4+m_1 = 3a \Rightarrow a = \frac{m_3+m_4+m_1}{3} > (m_3\,m_4\,m_1)^{1/3}$

$m_4+m_1+m_2 = 3b \Rightarrow b = \frac{m_4+m_1+m_2}{3} > (m_4\,m_1\,m_2)^{1/3}$

Hence $abcd > m_1\,m_2\,m_3\,m_4 \Rightarrow$ Result

Ex.77 If the polynomial $f(x) = 4x^4 - ax^3 + bx^2 - cx + 5$ where $a,b,c \in R$ has four positive real roots say r_1, r_2, r_3 and r_4, such that $\frac{r_1}{2}+\frac{r_2}{4}+\frac{r_3}{5}+\frac{r_4}{8} = 1$. Find the value of 'a'.

Sol. Consider 4 positive terms $\frac{r_1}{2}, \frac{r_2}{4}, \frac{r_3}{5}, \frac{r_4}{8}$ A.M. $= \frac{1}{4}\left(\frac{r_1}{2}+\frac{r_2}{4}+\frac{r_3}{5}+\frac{r_4}{8}\right) = \frac{1}{4}\times 1 = \frac{1}{4}$

G.M. $= \left(\frac{r_1}{2}\cdot\frac{r_2}{4}\cdot\frac{r_3}{5}\cdot\frac{r_4}{8}\right)^{1/4} = \left(\frac{r_1\cdot r_2\cdot r_3\cdot r_4}{2\cdot 4\cdot 5\cdot 8}\right)^{1/4}$ now, $r_1r_2r_3r_4 = \frac{5}{4}$

$\therefore$ G.M. $= \left[\frac{\cancel{5}}{4(2\cdot 4\cdot \cancel{5}\cdot 8)}\right]^{1/4} = \left(\frac{1}{2^8}\right)^{1/4} = \frac{1}{4}$. Hence A.M. = G.M. $\Rightarrow$ All numbers are equal

$\frac{r_1}{2} = \frac{r_2}{4} = \frac{r_3}{5} = \frac{r_4}{8} = k \quad \Rightarrow r_1 = 2k; \quad r_2 = 4k; \quad r_3 = 5k; \quad r_4 = 8k$

$\Rightarrow \prod r_1 = (2 \cdot 4 \cdot 5 \cdot 8)k^4 \quad \Rightarrow \quad \frac{5}{4} = (2 \cdot 4 \cdot 5 \cdot 8)k^4 \quad \therefore \quad k = 1/4$

hence $r_1 = \frac{1}{2}; \; r_2 = 1; \; r_3 = \frac{5}{4}; \; r_4 = 2 \quad \Rightarrow \quad \sum r_1 = \frac{19}{4}$

but $r_1 + r_2 + r_3 + r_4 = \frac{a}{4} \quad \Rightarrow \quad \frac{19}{4} = \frac{a}{4} \quad \Rightarrow \quad a = 19$

Ex.78 If $x > 0$ and $n \in N$, show that $\frac{x^n}{1 + x + x^2 + ... + x^{2n}} \le \frac{1}{1+2n}$.

Sol. $x^k + x^{-k} \ge 2, \quad k = 1, 2, 3, ..., n \Rightarrow 1 + \sum_{k=1}^{n} (x^k + x^{-k}) \ge 2n + 1$ Expand Σ and interpret

Ex.79 Prove that, $(a_1 b_1 + a_2 b_2 + ... + a_n b_n)^2 \le \left(a_1^2 + a_2^2 + ... + a_n^2\right) \left(b_1^2 + b_2^2 + ... + b_n^2\right)$ and the equality holds, only when $\frac{a_1}{b_1} = \frac{a_2}{b_2} = ... = \frac{a_n}{b_n}$. Also deduce that, Root Mean Square $\ge$ Arithmetic Mean.

Sol. Consider the quadratic $(a_1 x - b_1)^2 + (a_2 x - b_2)^2 + ... + (a_n - b_n x)^2 = 0$.

This can have real root only if each of the bracket vanishes i.e. $x = \frac{b_1}{a_1} = \frac{b_2}{a_2} = ... = \frac{b_n}{a_n}$

In all other cases D<0 i.e. $\left(a_1^2 + a_2^2 + ... + a_n^2\right) x^2 - 2(a_1 b_1 + a_2 b_2 + ... + a_n b_n) x + \left(b_1^2 + b_2^2 + ... + b_n^2\right) = 0$

$D < 0 \quad \Rightarrow \quad (a_1 b_1 + a_2 b_2 + ... + a_n b_n)^2 < \left(a_1^2 + a_2^2 + ... + a_n^2\right) \left(b_1^2 + b_2^2 + ... + b_n^2\right)$

equality hold when both roots are equal.

If $b_1 = b_2 = ... = b_n$ then $(a_1 + a_2 + ... + a_n)^2 < \left(a_1^2 + a_2^2 + ... + a_n^2\right) n$

$$\frac{a_1 + a_2 + ... + a_n}{n} < \sqrt{\frac{a_1^2 + a_2^2 + ... + a_n^2}{n}}$$

Ex.80 If a_1, a_2,......,a_n are n distinct odd natural numbers not divisible by any prime greater than 5, then prove that $\frac{1}{a_1}+\frac{1}{a_2}+...+\frac{1}{a_n}$ < 2.

Sol. Since each a_i is an odd number not divisible by a prime greater than 5, a_i can be written as $a_i = 3^r 5^s$ where r, s are non-negative integers. Thus, for all n $\in$ **N**.

$$\frac{1}{a_1}+\frac{1}{a_2}+...+\frac{1}{a_n} < \left(1+\frac{1}{3}+\frac{1}{3^2}+....\right)\left(1+\frac{1}{5}+\frac{1}{5^2}+....\right) = \left(\frac{1}{1-1/3}\right)\left(\frac{1}{1-1/5}\right) = \left(\frac{3}{2}\right)\left(\frac{5}{4}\right) = \frac{15}{8} < 2.$$

Ex.81 Let u_1, u_2, u_3, ..., u_n be an A.P. of positive terms with common difference d . Prove that

$$\sum_{i=1}^{n} u_i \geq n\sqrt{u_1^2+(n-1)\,d\,u_1}\,.$$

Sol. L H S = $S_n = \frac{n}{2}\,[\,2u_1 + (n-1)\,d\,]$

T P T $\left[u_1+\frac{(n-1)\,d}{2}\right]^2 \geq {u_1}^2 + (n-1)\,d\,u_1$ or $\frac{(n-1)^2\,d^2}{4} + u_1 d\,(n-1) \geq (n-1)\,d\,u_1$

or $(n-1)^2 d^2 \geq 0$ or $[(n-1)\,d]^2 \geq 0$ which is true , equally holds if $d = 0$

Ex.82 A_1, A_2,...., A_n are n A.M's, and H_1, H_2,....., H_n are n H.M's inserted between a and b. Prove that $\frac{A_1+A_n}{H_1+H_n}<\frac{A^2}{G^2}$, where A is the arithmetic mean and G is the geometric mean of a and b.

Sol. Since A_1, A_2,...., A_n are n arithmetic means between a and b,

$A_1 = \frac{na+b}{n+1}, A_n = \frac{a+nb}{n+1} \Rightarrow A_1 + A_n = a + b$ ...(1) Also, $\frac{1}{H_1}=\frac{a+nb}{(n+1)ab},\frac{1}{H_n}=\frac{an+b}{(n+1)ab}$ ....(2)

From (1) and (2)

$$\frac{A_1+A_n}{H_1+H_n}=\frac{a+b}{\frac{(n+1)ab}{a+nb}+\frac{(n+1)ab}{b+na}}=\frac{(b+na)(a+nb)(a+b)}{(n+1)ab(na+b+nb+a)}=\frac{(b+na)(a+nb)}{(n+1)^2ab}<\left(\frac{na+b+a+nb}{2}\right)^2\frac{1}{(n+1)^2ab}$$

$$=\frac{(a+b)^2}{4ab}=\frac{A^2}{G^2} \qquad \Rightarrow \qquad \frac{A_1+A_n}{H_1+H_n}<\frac{A^2}{G^2}.$$

Ex.83 If a , b , c are real and positive , prove that the inequality , $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} < \frac{a^8+b^8+c^8}{a^3\,b^3\,c^3}$.

Sol. This is equivalent to showing that , $a^8 + b^8 + c^8 > a^2 b^2 c^2 (ab + bc + ca)$

or $\frac{a^6}{b^2c^2} + \frac{b^6}{a^2c^2} + \frac{c^6}{a^2b^2} > ab + bc + ca$ — (1)

Now $\left(\frac{a^3}{bc} - \frac{b^3}{ac}\right)^2 > 0 \Rightarrow \frac{a^6}{b^2c^2} + \frac{b^6}{c^2a^2} > \frac{2a^2b^2}{c^2}$ Thus $\frac{a^6}{b^2c^2} + \frac{b^6}{a^2c^2} + \frac{c^6}{a^2b^2} > \frac{a^2b^2}{c^2} + \frac{b^2c^2}{a^2} + \frac{c^2a^2}{b^2}$

again $\left(\frac{ab}{c} - \frac{bc}{a}\right)^2 > 0 \Rightarrow \frac{a^2b^2}{c^2} + \frac{b^2c^2}{a^2} > 2b^2$ Thus $\frac{a^2b^2}{c^2} + \frac{b^2c^2}{a^2} + \frac{c^2a^2}{b^2} > a^2 + b^2 + c^2$

Finally it is well known that , $a^2 + b^2 + c^2 > ab + bc + ca$

Ex.84 Establish the inequality , $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots\ldots + \frac{1}{\sqrt{n}} > 2[\sqrt{n+1} - \sqrt{n}]$.

Sol. $\sqrt{r} + \sqrt{r} < \sqrt{r} + \sqrt{r+1}$ or $2\sqrt{r} < \sqrt{r} + \sqrt{r+1}$

or $\frac{1}{2\sqrt{r}} > \frac{1}{\sqrt{r+1}+\sqrt{r}}$ or $\frac{1}{2\sqrt{r}} > \sqrt{r+1} - \sqrt{r}$ or $\frac{1}{\sqrt{r}} > 2\left[\sqrt{r+1} - \sqrt{r}\right]$

Ex.85 If x, y, z be positive real numbers such that $x^2 + y^2 + z^2 = 27$, then show that $x^3 + y^3 + z^3 \geq 81$.

Sol. Applying Cauchy-Schwartz inequality to the two sets of numbers $x^{3/2}, y^{3/2}, z^{3/2}$, ; $x^{1/2}, y^{1/2}, z^{1/2}$

we have $(x^2 + y^2 + z^2)^2 \leq (x^3 + y^3 + z^3)(x + y + z)$. ...(1)

Again, applying Cauchy-Schwartz inequality to the two sets of numbers x, y, z; 1, 1, 1,

we have $(x + y + z)^2 \leq 3(x^2 + y^2 + z^2)$ ...(2)

Squaring both sides of (1), we have $(x^2 + y^2 + z^2)^4 \leq (x^3 + y^3 + z^3)^2 (x + y + z)^2$.

On using (2), the above inequality yields, $(x^2 + y^2 + z^2)^4 \leq 3(x^3 + y^3 + z^3)^2 (x^2 + y^2 + z^2)^2$ ...(3)

Since $x^2 + y^2 + z^2 = 27$. we have from (3), $(x^3 + y^3 + z^3)^2 \geq (81)^2$.

Taking positive square roots, we have $x^3 + y^3 + z^3 \geq 81$.

Ex.86 Prove that for any positive integer n > 1 the inequality $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots. + \frac{1}{\sqrt{n}} > 2(\sqrt{n+1} - \sqrt{n})$ holds true.

Sol. To prove this, reduce each term of the sum in the left-hand member:

$$\frac{1}{\sqrt{k}} > \frac{2}{\sqrt{k}+\sqrt{k+1}} = 2(\sqrt{k+1} - \sqrt{k})$$

Therefore, the left side of the inequality we want to prove can be reduced:

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots\ldots + \frac{1}{\sqrt{n}} > 2(\sqrt{2} - \sqrt{1}) + 2(\sqrt{3} - \sqrt{2}) + \ldots\ldots + 2(\sqrt{n} - \sqrt{n-1} + 2(\sqrt{n+1} - \sqrt{n})$$

Since the right side of this latter inequality is exactly equal to $2(\sqrt{n+1} - \sqrt{n})$, the original inequality is valid.

Ex.87 Prove that for every positive integer n the following inequality holds true: $\frac{1}{9}+\frac{1}{25}+.....+\frac{1}{(2n+1)^2}<\frac{1}{4}$

Sol. Noting that $\frac{2}{(2k+1)^2} < \frac{1}{2k} - \frac{1}{2k+2}$

we replace the sum in the left member of the inequality to be proved by the greater expression

$$\frac{1}{3^2}+\frac{1}{5^2} + ... + \frac{1}{(2n+1)^2} < \frac{1}{2}\left[\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{6}\right)+....+\left(\frac{1}{2n}-\frac{1}{2n+2}\right)\right]$$

However, this latter expression is equal to $\frac{1}{2}\left[\frac{1}{2}-\frac{1}{2n+2}\right] = \frac{1}{4} - \frac{1}{4n+4}$ and, obviously, is less than 1/4. Hence, the sum $\frac{1}{9}+\frac{1}{25}+....+\frac{1}{(2n+1)^2}$ is all the more so less than 1/4.

Ex.88 Prove that if a > 0, b > 0, then for any x and y the following inequality holds true;

$$a.2^x + b.3^y + 1 \le \sqrt{4^x+9^y+1} \cdot \sqrt{a^2+b^2+1}$$

Sol. By hypothesis, both sides of this inequlity are positive and so it is equivalent to the following:

$$(a.2^x + b.3^y + 1)^2 \le (4^x + 9^y + 1)\,(a^2 + b^2 + 1)$$

or $a^2.4^x + b^2 .\, 9^y + 1 + 2ab2^x3^y + 2a2^x + 2b3^y \le 4^xa^2 + 4^xb^2 + 4^x + 9^ya^2 + 9^yb^2 + 9^y + a^2 + b^2 + 1$

Transposing all terms of this inequality ot the right side, and then collecting like terms and regrouping, we can write it is the equivalent form $(a^29^y - 2ab2^x3^y + 4^xb^2) + (4^x - 2a2^x + a^2) + (9^y - 2b\,3^y + b^2) \ge 0$

Since each parenthesis is a perfect square, the original inequality is equivalent to the following obvious inequality : $(a3^y - b2^x)^2 + (2^x - a)^2 + (3^y - b)^2 \ge 0$. Hence the original inequality is true.

Note that this inequality is also true for any real values of a and b (the proof of this fact is left to the reader).

Ex.89 If a, b, c, d, e are positive real numbers, such that a + b +c + d + e = 8 and $a^2 + b^2 + c^2 + d^2 + e^2 = 16$, find the range of e.

Sol. As we know, $\left(\frac{a+b+c+d}{4}\right)^2 \le \frac{a^2+b^2+c^2+d^2}{4}$ ...(i) (using Tchebycheff's Inequality)

Where a +b + c + d + e = 8 and $a^2 + b^2 + c^2 + d^2 + e^2 = 16$

Fig.

$\therefore$ Equation (i) reduces to, $\left(\frac{8-e}{4}\right)^2 \le \frac{16-e^2}{4} \Rightarrow 64 + e^2 - 16e \le 4(16 - e^2) \Rightarrow 5e^2 - 16e \le 0$

$\Rightarrow$ e(5e – 16) $\le$ 0 {using number line rule} $\Rightarrow 0 \le e \le \frac{16}{5}$. Thus range fo e $\in \left[0, \frac{16}{5}\right]$